UNIVERSITÉ DE FRANCHE–COMTÉ

UFR DES SCIENCES ET TECHNIQUES

Thèse de Doctorat pour obtenir le grade de Docteur de l'Université de Franche–Comté

spécialité : MATHÉMATIQUES ET APPLICATIONS

Les aspects mathématiques des modèles de marchés financiers avec coûts de transaction

présentée et soutenue publiquement le **16 octobre 2013** par

Julien GRÉPAT

devant le jury composé de

Bruno Bouchard (Rapporteur), Université Paris–Dauphine

Clément Dombry (Examinateur), Université de Franche–Comté

Youri Kabanov (Directeur de Thèse), Université de Franche–Comté

Emmanuel Lépinette (Examinateur), Université Paris–Dauphine

Serge Pergamenchtchikov (Rapporteur), Université de Rouen

Bruno Saussereau (Examinateur), Université de Franche–Comté

“L’homme libre ne doit rien apprendre en esclave”

Platon

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Remerciements

Cette thèse s'est déroulée de septembre 2010 à septembre 2013 sous la direction du Professeur Youri Kabanov. Je lui suis reconnaissant pour son encadrement bienveillant, entre latitude et présence structurante.

Je remercie Bruno Bouchard et Serge Pergamenchtchikov qui m'ont fait l'honneur de rapporter cette thèse ainsi que Clément Dombry, Emmanuel Lépinette et Bruno Saussereau qui ont accepté de faire partie de mon jury.

Je remercie Emmanuel Lépinette pour notre collaboration qui a inspiré la présente troisième partie.

Je remercie le Conseil Régional de Franche–Comté pour le financement de mon contrat doctoral qui m'a permis d'être salarié pendant ces trois années.

Je remercie les membres du Laboratoire de Mathématiques de Besançon pour l'ambiance chaleureuse ainsi que l'équipe de l'ENSMM pour leur accueil lors de mes activités d'enseignement.

Résumé

Les marchés financiers occupent une place prépondérante dans l'économie. La future évolution des législations dans le domaine de la finance mondiale va rendre inévitable l'introduction de frictions pour éviter les mouvements spéculatifs des capitaux, toujours menaçants d'une crise. C'est pourquoi nous nous intéressons principalement, ici, aux modèles de marchés financiers avec coûts de transaction.

Cette thèse se compose de trois chapitres. Le premier établit un critère d'absence d'opportunité d'arbitrage donnant l'existence de systèmes de prix consistants, i.e. martingales évoluant dans le cône dual positif exprimé en unités physiques, pour une famille de modèles de marchés financiers en temps continu avec petits coûts de transaction.

Dans le deuxième chapitre, nous montrons la convergence des ensembles de sur-réplication d'une option européenne dans le cadre de la convergence topologique des ensembles. Dans des modèles multidimensionnels avec coûts de transaction décroissants à l'ordre $n^{-1/2}$, nous donnons une description de l'ensemble limite pour des modèles particuliers et en déduisons des inclusions pour les modèles généraux (modèles de KABANOV).

Le troisième chapitre est dédié à l'approximation du prix d'options européennes pour des modèles avec diffusion très générale (sans coûts de transaction). Nous étudions les propriétés des pay-offs pour pouvoir utiliser au mieux l'approximation du processus de prix du sous-jacent par un processus intuitif défini par récurrence grâce aux itérations de PICARD.

Introduction

La théorie des marchés financiers multi-actifs avec coûts de transaction permet une modélisation réaliste dans laquelle les objets, prix des actifs contingents, portefeuilles, stratégies, etc. sont à valeurs vectorielles. Deux visions se coordonnent : une expression en unités physiques des actifs sous-jacents ou leur cotation dans une monnaie de référence, le *numéraire*. Cette vectorialisation remplit le fossé entre, d'une part, les mathématiques financières traditionnelles qui ne considèrent qu'un actif en terme de numéraire et d'autre part, la vision mathématique intuitive de l'économie. Nous nous placerons dans des modèles pouvant être assimilés à des modèles de devises dont l'étude approfondie est l'objet du livre [24].

La notion de cône de solvabilité émerge immédiatement. Il s'agit de la partie de l'espace dont les positions permettent de ne plus avoir de dette sur aucun actif, en transférant de la richesse entre les coordonnées tout en s'acquittant des coûts de transaction. Cette notion est primordiale lors de la modélisation. Outre le fait qu'elle contraint les stratégies de portefeuille, elle détermine le cône dual positif. Celui-ci, exprimé en unités physiques, accueille, sous de bonnes hypothèses, des martingales qui ont une proximité avec le processus vectoriel de prix des actifs sous-jacents. Ces martingales sont nommées *systèmes de prix consistants* et permettent le calcul des prix de recouvrement des options européennes. En effet, le théorème de sur-réplication donné dans l'article [23] donne une caractérisation de tels capitaux initiaux. Il est basé sur la comparaison entre le niveau de richesse de ces derniers et l'espérance du prix de l'actif à répliquer dont la valeur est évaluée par les systèmes de prix consistants.

Cette thèse suit ce cheminement. Nous nous intéressons à un critère donnant l'existence de systèmes de prix consistants, puis travaillons sur les prix de sur-réplications de l'option européenne.

La première partie porte sur la théorie de l'arbitrage dans les modèles continus. L'opportunité d'arbitrage est la possibilité, par un portefeuille auto–finançant démarrant sans richesse initiale, d'arriver à une position solvable non nulle, i.e. d'engendrer des profits à coup sûr. L'hypothèse d'absence d'opportunité d'arbitrage dans les modèles de marchés financiers est communément acceptée dans le monde de la finance, tant par les praticiens que par les théoriciens. En effet, les arbitragistes font disparaître toute opportunité d'arbitrage quasi instantanément, et par conséquent les modélisations n'en tiennent pas compte. Cette théorie de l'absence d'arbitrage a pour but de donner l'existence de systèmes de prix consistants.

Dans les modèles discrets sans friction, il s'agit du théorème de Dalang–Merton–Willinger [7], donnant l'existence d'une probabilité équivalente sous laquelle le prix est une martingale. Dans le cadre des coûts de transaction, la théorie s'est développée autour d'une adaptation probabiliste du théorème de séparation d'Hahn–Banach. En effet, la martingale considérée est le processus d'espérance conditionnelle d'une variable aléatoire qui sépare l'ensemble des valeurs terminales de portefeuille (à capital initial nul) de l'ensemble des variables aléatoires à coordonnées positives. Nous citerons en particulier le théorème pour le critère d'absence d'arbitrage robuste NA^r établi dans les articles [27, 33].

Dans les modèles continus, quantité de ces théorèmes ne se généralisent pas. Par conséquent des hypothèses plus fortes du type "No Free Lunch" (NFL-NFLVR-NFLBR) sont introduites et étudiées dans les différentes versions du "théorème fondamental de l'évaluation d'actif" (F.T.A.P.). Dans l'article [9], l'existence de martingales est montrée dans le cadre sans coûts de transaction, et plus récemment avec friction dans [10].

Nous proposons un critère simple d'absence d'arbitrage qui a l'avantage de s'exprimer de manière analogue avec le temps discret. La caractérisation de l'hypothèse d'absence d'arbitrage concerne toute une famille de modèles continus avec coûts de transaction, et donne l'existence de systèmes de prix consistants pour la famille de modèles. Ce résultat repose sur une discrétisation de l'intervalle de temps par des temps d'arrêts. Sur ces suites, nous appliquons le théorème d'absence d'arbitrage NA^r discret et une étude grâce à la convergence presque sûre permet l'extension au temps continu.

Le second chapitre est une étude de la limite des ensembles de sur-réplication d'une option européenne dans une suite de modèles multidimensionnels discrets avec coûts de transaction tendant vers zéro avec le pas de temps. Depuis la thèse de Bachelier, "Théorie de la Spéculation" (1900) et les formules de Black-Scholes [2], le paradigme de la finance est continu alors qu'en réalité, les actualisations se font le long d'une grille de dates discrètes prédéfinie. Le lien entre ces deux mondes est nécessaire et mène à certains paradoxes. Il est bien connu que l'observation discrète des évolutions du prix du sous-jacent (de plus en plus fréquente) n'entraîne pas la convergence du prix de l'option vers le prix théorique du modèle continu. L'idée naissante des travaux de Black–Scholes et de Leland [31] est qu'une certaine friction est implicite dans les marchés. C'est ainsi que la convergence du prix de l'option est prouvée, dans les modèles de Leland–Lott, du discret vers le continu, grâce à l'introduction de coûts de transaction décroissants.

Nous nous intéressons, comme dans Kusuoka [30], à la limite des prix de sur-réplication d'une option européenne "étendue" dans des suites de modèles discrets où les coûts de transaction décroissent à l'ordre $n^{-1/2}$. Les prix des sous-jacents sont modélisés par des processus très simples basés sur des marches aléatoires qui convergent en loi vers un mouvement Brownien géométrique. De manière étonnante, dans le modèle limite, il faut évaluer l'option non pas sur l'unique système de prix consistant du modèle complet conduit par le mouvement Brownien géométrique, mais par rapport à un ensemble de martingales au "comportement" proche dudit système de prix consistants.

Nous considérons des modèles de marchés multidimensionnels et regardons la convergence de tout l'ensemble de sur-réplication généré par chacun des modèles de la suite, étendant ainsi le résultat dans [30]. Dans des modèles simplifiés, nous regardons la limite des ensembles de sur-réplication dans le cadre de la topologie des fermés de $\mathbb{R}^{1+d}$, voir [1, 20], topologie qui coincide avec la célèbre topologie de Hausdorff sur les compacts. Le théorème limite s'appuie sur le théorème de sur-réplication qui donne une représentation de ces ensembles par les systèmes de prix consistants. Ainsi, une démarche "duale" nous fait utiliser la théorie de la convergence faible sur l'espace de Skorohod détaillée dans les livres [3, 22]. La convergence faible des systèmes de prix consistants permet en effet de montrer que le critère de sur-réplication est vérifié pour l'ensemble des vecteurs limites. Les ramifications concernant des modèles plus généraux suivent.

Dans le troisième chapitre, nous proposons une approximation du prix d'options à pay-off assez général, sans coûts de transaction, grâce à une approximation du prix du sous-jacent. Ce prix peut être conduit par un processus avec diffusion très générale. Une erreur théorique est calculée grâce à l'erreur quadratique moyenne de l'approximation.

Les formules de Black–Scholes [2] sont le point de départ des méthodes de recouvrement de l'option européenne. Dans ce marché, l'évolution du prix du sous-jacent est supposée suivre un mouvement Brownien géométrique, où la volatilité est constante. Malheureusement les tests statistiques rejettent ces modèles et des processus de diffusion plus élaborés apparaissent. Différents modèles avec volatilité locale sont étudiés, les volatilités smiles et skew [11, 5], les volatilités stochastiques [21], etc.

Avec la complexité des équations différentielles stochastiques générées par ces modèles, des simulations numériques s'imposent. Dans [4, 29], on utilise la méthode de Monte-Carlo pour des schémas aux différences finies. En particulier le schéma d'Euler, intuitif et simple, souffre d'un manque de précision et converge, sans hypothèses fortes, à l'ordre $n^{-1/2}$. Les schémas plus rapides deviennent, quant à eux, bien moins transparents.

Dans les articles [13, 14], des schémas utilisant les itérations de Picard permettent d'approcher le processus de diffusion par étapes successives grâce à une fonction déterministe. Cette méthode est suivie d'un développement en polynômes d'Hermite et de l'itération d'intégrales stochastiques grâce à la formule de chaos de Wiener–Ito. Cependant il n'est pas clair que les termes au-delà des trois considérés soient aisément calculables et de fait, l'acuité de l'approximation est illustrée par des simulations numériques mais n'est pas théoriquement calculée.

Nous proposons, à la suite des itérations de Picard, une approximation des diffusions successives grâce à la discrétisation du mouvement Brownien. Nous sommes alors confrontés à la difficulté du manque de précision de l'approximation. Lorsque nous arrêtons les itérations à l'ordre 2, le résultat tient son intérêt du fait que le processus approximant reste continu pour éviter la perte de vitesse théorique d'ordre $n^{-1/2}$ due à la discrétisation de l'intégrale stochastique. Puisque les pay-offs de l'option "européenne" (généralisée) ne nécessitent que le calcul à certaines dates discrètes, il est possible d'utiliser des simulations type Monte–Carlo. Par conséquent ce schéma, aussi simple que le schéma d'Euler, offre une vitesse de convergence plus rapide que pour ce dernier sans hypothèse restrictive, dès lors que l'on accepte une erreur systématique dans l'esprit de celle des schémas de [13, 14].

Dans le cas où nous considérons davantage d'itérations, notre approximation est de l'ordre $n^{-1/2}$, perdant l'erreur systématique précédente. Cette partie d'étude s'ouvre sur plusieurs horizons de recherche, en particulier sur la question d'une approximation plus élaborée du processus de diffusion itéré pour obtenir de meilleures vitesses de convergence ou encore sur l'approximation d'autres processus tel le modèle C.I.R..

Notations

Thoughout the text, we shall use the following notations.

- for a vector $v = (v^0, v^1, \cdots, v^d) \in \mathbb{R}^{1+d}$

$$|v| := \max_{0 \le i \le d} |v^i|;$$

- for vectors $v, w \in \mathbb{R}^{1+d}$

$$vw = \sum_{i=0}^{d} v^i w^i;$$

- the canonical vectors of $\mathbb{R}^d$ are denoted by e^i and $\mathbf{1} := (1, \ldots, 1)$.

- the max-norm ball of radius ε with center at $\mathbf{1} := (1, \ldots, 1)$ is denoted by $\mathbf{1} + U_\varepsilon$, where

$$U_\varepsilon := \{x \in \mathbf{R}^d : \max_i |x^i| \le \varepsilon\},$$

- for a matrix $c = (c^{ij})_{1 \le i,j \le d}$

$$c^i := \begin{pmatrix} c^{i1} & \cdots & c^{id} \end{pmatrix}$$

  and the notation c' stands for the transposed matrix;

- for a sequence of random variables $(\xi^n)_{n \in \mathbb{N}}$ the symbol $O(n^{-a})$ means that there exists a positive constant κ such that $n^a |\xi^n| \le \kappa$ a.s. for any n;

- $\mathbb{D}(\mathbb{R}^d)$ is the Skorohod space of the cádlág functions $x : [0,T] \to \mathbb{R}^d$ while $\mathbb{C}(\mathbb{R}^d)$ denotes the space of continuous functions taking values in $\mathbb{R}^d$ with the uniform norm

$$||x||_T = \sup_{t \leq T} |x_t|.$$

- for a process H, we write in short

$$H \cdot W_t := \int_0^t H_u dW_u;$$

- for a random variable ζ, we set the L^p-norm

$$||\zeta||_p = (E|\zeta|^p)^{1/p}.$$

For a sake of simplicity, we use the following abuse of notation: from line to line, a constant K, κ or C may designate different constants which are independent of any variables except, may be, fixed parameters of the problem like the maturity date T for instance. Otherwise, we may use the notation C_m when the constant C_m depends on a parameter m but may also change from line to line.

Part I

Arbitrage Theory

The arbitrage theory for financial markets with proportional transaction costs is one of the most advanced and interesting domains of mathematical finance. It success is due to a geometric viewpoint which provides an appropriate language to attack problems. The approach based on convex geometry not only makes arguments much more transparent comparatively with traditional, "parametric", modeling but also allows to put problems in a more general mathematical framework. To the date, for the discrete-time setting there is a plethora of criteria for various types of arbitrage, see Chapter 3 of the book [24]. In a surprising contrast, for continuous-time models only a few results on the no-arbitrage criteria are available. In the recent paper [19] Guasoni, Rásonyi, and Schachermayer established an interesting result in this direction. They formulated the question on sufficient and necessary conditions for the absence of arbitrage not for a single model but for a whole family of them. Namely, they considered two-asset models with a fixed continuous price process and constant transaction costs tending to zero. In a rather spectacular way, the resulting no-arbitrage criterion happens to be very simple: the NA^w-property holds for each model if and only if each model admits a consistent price system. The advantage of such a formulation is clear: topological properties, common in this theory, are not involved. It looks very similar to the no-arbitrage criterion for the model with finite Ω, see Th. 3.1.1 in the book [24] and Th. 3.2 in the original paper [25].

Apparently, this result merits to be put in the mainstream of the theory of financial markets with transaction costs. In the present note we extend, using the now "standard" geometric approach, the main theorem of [19] to the case of multi-asset models. The paper [19] serves us as the roadmap.

Cette partie est issue de l'article [17], coécrit avec Youri Kabanov.

Chapter 1

The Main Result

1.1 Main Result

Let $\varepsilon \in]0,1]$ and let $K^{\varepsilon *} := \mathbf{R}_+(\mathbf{1} + U_\varepsilon)$, where we recall the notation $U_\varepsilon := \{x \in \mathbf{R}^d : \max_i |x^i| \leq \varepsilon\}$. That is, $K^{\varepsilon *}$ is the closed convex cone in $\mathbf{R}^d$ generated by the max-norm ball of radius ε with center at $\mathbf{1} := (1, \dots, 1)$. We denote by K^ε the (positive) dual cone of $K^{\varepsilon *}$.

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ be a stochastic basis and let $S = (S_t)_{t \leq T}$ be a **continuous** semimartingale with strictly positive components. We consider the linear controlled stochastic equation

$$dV_t^i = V_{t-}^i dY_t^i + dB_t^i, \quad V_0^i = 0, \qquad i \leq d,$$

where Y^i is the stochastic logarithm of S^i, i.e. $dY_t^i = dS_t^i / S_t^i$, $Y_0^i = 1$, and the strategy B is a predictable càdlàg process of bounded variation with $\dot{B} \in -K^\varepsilon$. The notation $\dot{B}$ stands for (a measurable version of) the Radon–Nikodym derivative of B with respect to $||B||$, the total variation process of B.

A strategy B is ε-*admissible* if for the process $V = V^B$ there is a constant κ such that $V_t + \kappa S_t \in K^\varepsilon$ for all $t \leq T$. The set of processes V corresponding to ε-admissible strategies is denoted by $A_0^{T\varepsilon}$ while the notation $A_0^{T\varepsilon}(T)$ is reserved for the set of random variables V_T, $V \in A_0^{T\varepsilon}$.

Using the random operator

$$\phi_t : (x^1, ..., x^d) \mapsto (x^1/S_t^1, ..., x^d/S_t^d)$$

define the random cone $\widehat{K}_t^\varepsilon = \phi_t K^\varepsilon$ with the dual $\widehat{K}_t^{\varepsilon *} = \phi_t^{-1} K^{\varepsilon *}$. Put $\widehat{V}_t = \phi_t V_t$. We denote by $\mathcal{M}_0^T(\widehat{K}^{\varepsilon *} \setminus \{0\})$ the set of martingales Z such that $Z_t \in \widehat{K}_t^{\varepsilon *} \setminus \{0\}$ for all $t \leq T$.

Theorem 1.1.1 *We have:*

$$A_0^{T\varepsilon}(T) \cap L^0(\mathbf{R}_+^d, \mathcal{F}_T) = \{0\} \ \ \forall \varepsilon \in]0,1] \quad \Leftrightarrow \quad \mathcal{M}_0^T(\widehat{K}^{\varepsilon *} \setminus \{0\}) \neq \emptyset \ \ \forall \varepsilon \in]0,1].$$

The strategy of the proof of Theorem 1.1.1.

To prove the nontrivial implication ($\Rightarrow$) we exploit the fact that the universal NA^w-property holds for any imbedded discrete-time model. Using the criterion for NA^r-property we deduce from here in Section 1.2 the existence of a "universal chain", that is there exists a sequence of stopping times τ_n increasing stationary to T and such that $\mathcal{M}_0^{\tau_n}(\widehat{K}^{\varepsilon *} \setminus \{0\}) \neq \emptyset$ for all $\varepsilon \in]0,1]$ and $n \geq 1$. In an analogy with [19], we relate with this "universal chain" functions $F^i(\varepsilon)$, $i \leq d$, and check that there is, for each i, an alternative: either $F^i = 0$, or $F^i(0+) = 1$. This is the most involved part of the proof isolated in Section 1.3. If all $F^i = 0$, the sets $\mathcal{M}_0^{\tau_n}(\widehat{K}^{\varepsilon *} \setminus \{0\})$ are non-empty and we conclude. If there is a coordinate i for which $F^i(0+) = 1$, there exists a strict arbitrage opportunity, see Section 1.4. In Section 2.2 we discuss the properties of richness of the set of consistent price systems.

1.2 Universal Discrete-Time NA^w-property

We say that the continuous-time model has *universal discrete-time NA^w-property* if for any $\varepsilon > 0$, $N \geq 2$, and an increasing sequence of stopping times $\sigma_1, \ldots, \sigma_N$ with values in $[0,T]$ and such that $\sigma_n < \sigma_{n+1}$ on the set $\{\sigma_n < T\}$, we have that

$$L^0(\mathbf{R}_+^d, \mathcal{F}_T) \cap \sum_{n=1}^{N} L^0(-\phi_{\sigma_n} K^\varepsilon, \mathcal{F}_{\sigma_n}) = \{0\}.$$

Proposition 1.2.1 *Suppose that the model has the universal discrete-time NA^w-property. Then there is a strictly increasing sequence of stopping times τ_n with $P(\tau_n < T) \to 0$ as $n \to \infty$ such that for any N and $\varepsilon \in]0,1]$ the set $\mathcal{M}_0^{\tau_N}(\widehat{K}^{\varepsilon *} \setminus \{0\})$ is non-empty.*

Proof. Define recursively the increasing sequence of stopping times:

$$\begin{aligned}
\sigma_0 &= 0, \\
\sigma_n &= \sigma_n^\varepsilon := \inf\{t \geq \sigma_{n-1} : \max_{i \leq d} |\ln S_t^i - \ln S_{\sigma_{n-1}}^i| \geq \ln(1+\varepsilon/8)\},
\end{aligned}$$

for $n \geq 1$. This sequence has the following property which we formulate as a lemma.

Lemma 1.2.2 *For any integer $N \geq 1$ there exists $Z \in \mathcal{M}_0^{\sigma_N}(\widehat{K}^{\varepsilon *} \setminus \{0\})$.*

Proof. To avoid a new notation we suppose without loss of generality that $S = S^{\sigma_N}$. Let $X_n := S_{\sigma_n}$. By our assumption and in virtue of the criterion for the NA^r-property there is a discrete-time martingale $(M_n)_{n \leq N}$ with $M_n \in L^\infty(\phi_{\sigma_n}^{-1} K^{\varepsilon/4*} \setminus \{0\})$, see Th. 3.2.1 in [24] or Th. 3 in [27]. Put $Z_t := E(M_N|\mathcal{F}_t)$ and $\tilde{Z}_t := \phi_t Z_t$. Let us check that $Z \in \mathcal{M}_0^{\sigma_N}(\widehat{K}^{\varepsilon *} \setminus \{0\})$. On the set $\{t \in [\sigma_{n-1}, \sigma_n]\}$

$$\tilde{Z}_t = E(\phi_t \phi_{\sigma_n}^{-1} \tilde{Z}_{\sigma_n} | \mathcal{F}_t).$$

Note that

$$(1+\varepsilon/8)^{-2} \leq \frac{S_{\sigma_n}^i}{S_t^i} = \frac{S_{\sigma_{n-1}}^i}{S_t^i} \frac{S_{\sigma_n}^i}{S_{\sigma_{n-1}}^i} \leq (1+\varepsilon/8)^2.$$

Therefore,

$$(1+\varepsilon/8)^{-2} E(\tilde{Z}_{\sigma_n}^i | \mathcal{F}_t) \leq \tilde{Z}_t^i \leq (1+\varepsilon/8)^2 E(\tilde{Z}_{\sigma_n}^i | \mathcal{F}_t).$$

But $E(\tilde{Z}_{\sigma_n}|\mathcal{F}_t) = E(\phi_{\sigma_n} M_n | \mathcal{F}_t) \in \operatorname{cone}(\mathbf{1} + U_{\varepsilon/4}) \setminus \{0\}$, i.e. the components of $E(\tilde{Z}_{\sigma_n}|\mathcal{F}_t)$ take values in the interval with the extremities $\lambda(1 \pm \varepsilon/4)$ where $\lambda > 0$ depends on n and ω. Thus,

$$1 - \varepsilon \leq (1+\varepsilon/8)^{-2}(1-\varepsilon/4) \leq \tilde{Z}_t^i/\lambda \leq (1+\varepsilon/8)^2(1+\varepsilon/4) \leq 1+\varepsilon.$$

This implies the assertion of the lemma. □

To finish the proof of the proposition, we proceed exactly as at the end of proof of Th. 1.4 in [19]. Namely, we take a sequence of $\varepsilon_k \downarrow 0$. For each $n \geq 1$ we find an integer $N_{n,k}$ such that

$$P(\sigma^{\varepsilon_k}_{N_{n,k}} < T) < 2^{-(n+k)}.$$

Without loss of generality we assume that for each k the sequence $(N_{n,k})_{n\geq 1}$ is increasing. The increasing sequence of stopping times $\tau_n := \min_{k\geq 1} \sigma^{\varepsilon_k}_{N_{n,k}}$ converges to T stationary: $P(\tau_n < T) \leq 2^{-n}$. Applying the lemma with ε_k we obtain that for the process S stopped at $\sigma^{\varepsilon_k}_{N_{n,k}}$ there exists an ε_k-consistent price system. The latter, being stopped at τ_n, is an ε_k-consistent price system for S^{τ_n}. □

We call the sequence (τ_n) which existence was established above *universal chain.*

1.3 Properties of Universal Chains

We explore properties of a universal chain assuming that $P(\tau_n < T) > 0$ for all n.

Let us introduce the set $\mathcal{T}_T$ of stopping times σ such that $P(\sigma < T) > 0$ and, for some n, the inequality $\sigma \leq \tau_n$ holds on $\{\sigma < T\}$. This set is non empty: by the adopted hypothesis it contains all τ_n.

Let $\sigma \in \mathcal{T}_T$ and let n be such that $\sigma \leq \tau_n$ holds on $\{\sigma < T\}$.

We denote by $\mathcal{M}^i(\sigma, \varepsilon, n)$ the set of processes Z such that:

(i) $Z = 0$ on $\{\sigma = T\}$;

(ii) Z is a martingale on $[\sigma, \tau_n]$, i.e. $E(Z_{\tau_n}|\mathcal{F}_\vartheta) = Z_\vartheta$ for any stopping time ϑ such that $\sigma \leq \vartheta \leq \tau_n$ on $\{\sigma < T\}$;

(iii) $Z_t(\omega) \in \operatorname{int} \widehat{K}^{\varepsilon*}_t(\omega)$ when $\sigma(\omega) < T$ and $t \in [\sigma(\omega), \tau_n(\omega)]$;

(iv) $EZ^i_\sigma I_{\{\sigma<T\}} = 1$.

Note that the process $Z = \tilde{Z} I_{\{\sigma<T\}}/E\tilde{Z}^i_\sigma I_{\{\sigma<T\}}$ belongs to $\mathcal{M}^i(\sigma,\varepsilon,n)$ provided that $\tilde{Z} \in \mathcal{M}^{\tau_n}_0(\operatorname{int}\widehat{K}^{\varepsilon *})$.

Let $F^i(\varepsilon) := \sup_{\sigma\in\mathcal{T}_T} F^i(\sigma,\varepsilon)$ where

$$F^i(\sigma,\varepsilon) := \varlimsup_n \inf_{Z\in\mathcal{M}^i(\sigma,\varepsilon,n)} EZ^i_{\tau_n} I_{\{\tau_n<T\}}.$$

We also put

$$f^i(\sigma,\varepsilon,n) := \operatorname{ess}\inf_{Z\in\mathcal{M}^i(\sigma,\varepsilon,n)} E((Z^i_{\tau_n}/Z^i_\sigma) I_{\{\tau_n<T\}}|\mathcal{F}_\sigma).$$

Lemma 1.3.1 *For any $Z \in \mathcal{M}^i(\sigma,\varepsilon,n)$ there is a process $\bar{Z} \in \mathcal{M}^i(\sigma,\varepsilon,n+1)$ such that $\bar{Z}^{\tau_n} = Z^{\tau_n}$.*

Proof. To explain the idea we suppose first that $Z \in \mathcal{M}^i(\sigma,\varepsilon',n)$ for some $\varepsilon' < \varepsilon$. Take $\delta > 0$ and a process $\tilde{Z} \in \mathcal{M}^i(\sigma,\delta,n+1)$. Define the process $\bar{Z}$ with components

$$\bar{Z}^j := Z^j I_{[0,\tau_n[} + \frac{Z^j_{\tau_n}}{\tilde{Z}^j_{\tau_n}} \tilde{Z}^j I_{[\tau_n,T]}.$$

Note that

$$\phi_t Z_t = \lambda_t(1+u^1_t,\dots,1+u^d_t), \quad t\in[\sigma,\tau_n],$$
$$\phi_t \tilde{Z}_t = \tilde{\lambda}_t(1+\tilde{u}^1_t,\dots,1+\tilde{u}^d_t), \quad t\in[\tau_n,\tau_{n+1}],$$

where $\max_j |u^j| \le \varepsilon'$, $\max_j |\tilde{u}^j| \le \delta$ and $\lambda_t, \tilde{\lambda}_t > 0$. It follows that $\bar{Z}$ belongs to $\mathcal{M}^i(\sigma,\bar{\varepsilon},n+1)$ with

$$\bar{\varepsilon} = \frac{(1+\varepsilon')(1+\delta)}{1-\delta} - 1.$$

Since $\bar{\varepsilon} < \varepsilon$ for sufficiently small $\delta = \delta(\varepsilon')$, the result follows.

In the general case we consider the partition of the set $\{\sigma < T\}$ on $\mathcal{F}_{\tau_n}$-measurable subsets A_k, on which the process Z evolves, on the interval $[\sigma,\tau_n]$, in the cones $\widehat{K}^{\varepsilon_k *}$, where $\varepsilon_k := (\varepsilon - 1/k)\vee 0$. As above, take processes $\tilde{Z}^k \in \mathcal{M}^i(\sigma,\delta_k,n+1)$ with $\delta_k = \delta(\varepsilon_k)$. Then the process $\bar{Z}$ with components

$$\bar{Z}^j := Z^j I_{[0,\tau_n[} + \sum_k \frac{Z^j_{\tau_n}}{\tilde{Z}^{kj}_{\tau_n}} \tilde{Z}^{kj} I_{A_k} I_{[\tau_n,T]}$$

meets the requirements. □

Lemma 1.3.2 *The sequence* $(f^i(\sigma,\varepsilon,n))_{n\geq 0}$ *is decreasing and its limit* $f^i(\sigma,\varepsilon) \leq F^i(\varepsilon)$.

Proof. By Lemma 1.3.1 for any $Z \in \mathcal{M}^i(\sigma,\varepsilon,n)$ there is a process $\bar{Z} \in \mathcal{M}^i(\sigma,\varepsilon,n+1)$ such that $\bar{Z}^{\tau_n} = Z^{\tau_n}$. Using the martingale property of $\bar{Z}$ we get that

$$\begin{aligned} E((Z^i_{\tau_n}/Z^i_\sigma)I_{\{\tau_n<T\}}|\mathcal{F}_\sigma) &= E((\bar{Z}^i_{\tau_n}/\bar{Z}^i_\sigma)I_{\{\tau_n<T\}}|\mathcal{F}_\sigma) \\ &\geq E((\bar{Z}^i_{\tau_{n+1}}/\bar{Z}^i_\sigma)I_{\{\tau_{n+1}<T\}}|\mathcal{F}_\sigma). \end{aligned}$$

It follows that $f^i(\sigma,\varepsilon,n) \geq f^i(\sigma,\varepsilon,n+1)$.

Suppose that the claimed inequality $f^i(\sigma,\varepsilon) \leq F^i(\varepsilon)$ fails. Then there exist a non-null $\mathcal{F}_\sigma$-measurable set $A \subseteq \{\sigma < T\}$ and a constant $a > 0$ such that for all sufficiently large n

$$f^i(\sigma,\varepsilon,n)I_A \geq (F^i(\varepsilon)+a)I_A.$$

Define the stopping time $\sigma_A := \sigma I_A + T I_{A^c}$ and note that for any $Z \in \mathcal{M}^i(\sigma,\varepsilon,n)$ the process ZI_A/EZI_A is an element of $\mathcal{M}^i(\sigma_A,\varepsilon,n)$. Since $E(\xi|\mathcal{F}_\sigma)I_A = E(\xi|\mathcal{F}_{\sigma_A})I_A$, we have the bound

$$f^i(\sigma_A,\varepsilon,n)I_A \geq f^i(\sigma,\varepsilon,n)I_A.$$

Thus, for any $Z \in \mathcal{M}^i(\sigma_A,\varepsilon,n)$ and sufficiently large n

$$EZ^i_{\tau_n}I_{\{\tau_n<T\}} = EZ^i_{\sigma_A}E((Z^i_{\tau_n}/Z^i_{\sigma_A})I_{\{\tau_n<T\}}|\mathcal{F}_{\sigma_A}) \geq F^i(\varepsilon)+a$$

in contradiction with the definition of $F^i(\varepsilon)$. $\Box$

Lemma 1.3.3 *Let* $\sigma \in \mathcal{T}_T$ *be such that* $\sigma \leq \tau_{n_0}$ *on the set* $\{\sigma < T\}$ *and let* $\varepsilon,\delta > 0$. *Then there are* $n \geq n_0$, $\Gamma \in \mathcal{F}_\sigma$ *with* $P(\Gamma) \leq \delta$, *and* $Z \in \mathcal{M}^i(\sigma,\varepsilon,n)$ *such that* $Z^i_\sigma = \eta := I_{\{\sigma<T\}}/EI_{\{\sigma<T\}}$ *and*

$$E(Z^i_{\tau_n}I_{\{\tau_n<T\}}|\mathcal{F}_\sigma) \leq \frac{I_{\{\sigma<T\}}}{EI_{\{\sigma<T\}}}[(F^i(\varepsilon)+\delta)I_{\Gamma^c}+I_\Gamma].$$

Proof. Recall that the essential infimum ξ of a family of random variables $\{\xi^\alpha\}$ is the limit of a decreasing sequence of random variables of the form $\xi^{\alpha_1} \wedge \xi^{\alpha_2} \wedge ... \wedge \xi^{\alpha_m}$, $m \to \infty$. Thus, for any $a > 0$ the sets $\{\xi^{\alpha_k} \le \xi + a\}$ form a covering of Ω. Using the standard procedure, one can construct from this covering a measurable partition of Ω by sets A^k such that $\xi^{\alpha_k} \le \xi + a$ on A^k.

Thus, for any fixed $n \ge n_0$ there are a countable partition of the set $\{\sigma < T\}$ into $\mathcal{F}_\sigma$-measurable sets $A^{n,k}$ and a sequence of $Z^{n,k} \in \mathcal{M}^i(\sigma, \varepsilon, n)$ such that

$$E((Z^{n,k,i}_{\tau_n}/Z^{n,k,i}_\sigma)I_{\{\tau_n<T\}}|\mathcal{F}_\sigma) \le f^i(\sigma, \varepsilon, n) + 1/n \quad \text{on } A^{n,k}.$$

Put, for $t \in [\sigma, \tau_n]$,

$$\tilde{Z}^n_t := \eta \sum_{k=1}^{\infty} \frac{1}{Z^{n,k,i}_\sigma} Z^{n,k}_t I_{A^{n,k}}.$$

Then $\tilde{Z}^n \in \mathcal{M}^i(\sigma, \varepsilon, n)$, $\tilde{Z}^{n,i}_\sigma = \eta$, and

$$E(\tilde{Z}^{n,i}_{\tau_n} I_{\{\tau_n<T\}}|\mathcal{F}_\sigma) = \eta E((\tilde{Z}^{n,i}_{\tau_n}/\eta)I_{\{\tau_n<T\}}|\mathcal{F}_\sigma) \le \frac{I_{\{\sigma<T\}}}{EI_{\{\sigma<T\}}}[f^i(\sigma, \varepsilon, n) + 1/n].$$

Note that $f^i(\sigma, \varepsilon, n) + 1/n$ decreases to $f^i(\sigma, \varepsilon) \le F^i(\varepsilon)$. By the Egorov theorem the convergence is uniform outside of a set Γ of arbitrary small probability. The assertion of the lemma follows from here immediately. $\square$

Proposition 1.3.4 *For any $\varepsilon_1, \varepsilon_2$ we have the inequality*

$$F^i(\varepsilon_1)F^i(\varepsilon_2) \ge F^i((1+\varepsilon_1)(1+\varepsilon_2)/(1-\varepsilon_2) - 1). \tag{1.3.1}$$

Either $F^i = 0$, or there is $c^i \ge 0$ such that $F^i(\varepsilon) \ge e^{-c^i \varepsilon^{1/3}}$ for all sufficiently small ε.

Proof. Fix $\delta > 0$ and a stopping time $\sigma \le \tau_{n_0}$ on the set $\{\sigma < T\}$. According to the above lemma there are $n \ge n_0$ and $Z^1 \in \mathcal{M}^i(\sigma, \varepsilon_1, n)$ such that

$$EZ^{1i}_{\tau_n} I_{\{\tau_n<T\}} \le F^i(\varepsilon_1) + \delta.$$

Using the same lemma again (but now with τ_n playing the role of σ), we find $m > n$ and $Z^2 \in \mathcal{M}^i(\tau_n, \varepsilon_2, m)$ with $Z^{2i}_{\tau_n} = I_{\{\tau_n<T\}}/EI_{\{\tau_n<T\}}$ such that outside of a set $\Gamma \in \mathcal{F}_{\tau_n}$ with $P(\Gamma) \le \delta$ we have the bound

$$E(Z^{2i}_{\tau_m} I_{\{\tau_m<T\}}|\mathcal{F}_{\tau_n}) \le \frac{I_{\{\tau_n<T\}}}{EI_{\{\tau_n<T\}}}[(F^i(\varepsilon_2)+\delta)I_{\Gamma^c} + I_\Gamma].$$

Define on $[\sigma, \tau_m]$ the martingale Z with $Z^j_t := Z^{1j}_t$ on $[\sigma, \tau_n]$ and $Z^j_t := Z^{2j}_t Z^{1j}_{\tau_n}/Z^{2j}_{\tau_n}$ on $[\tau_n, \tau_m]$, $j = 1, \dots, d$. Then

$$\phi_t Z^1_t = \lambda^1_t(1+u^{11}_t, \dots, 1+u^{1d}_t), \quad t \in [\sigma, \tau_n],$$

$$\phi_t Z^2_t = \lambda^2_t(1+u^{21}_t, \dots, 1+u^{2d}_t), \quad t \in [\tau_n, \tau_m],$$

where $\max_j |u^{1j}| \le \varepsilon_1$, $\max_j |u^{2j}| \le \varepsilon_2$ and $\lambda^1_t, \lambda^2_t > 0$. It follows that

$$Z \in \mathcal{M}^i(\sigma, (1+\varepsilon_1)(1+\varepsilon_2)/(1-\varepsilon_2) - 1, m).$$

Note also that

$$\begin{aligned} EZ^i_{\tau_m} I_{\{\tau_m<T\}} &= P(\tau_n < T) EZ^{2i}_{\tau_m} Z^{1i}_{\tau_n} I_{\{\tau_m<T\}} \\ &\le P(\tau_n < T) EZ^{1i}_{\tau_n} I_{\{\tau_n<T\}} E(Z^{2i}_{\tau_m} I_{\{\tau_m<T\}}|\mathcal{F}_{\tau_n}). \end{aligned}$$

Hence,

$$EZ^i_{\tau_m} I_{\{\tau_m<T\}} \le (F^i(\varepsilon_1)+\delta)(F^i(\varepsilon_2)+\delta) + EZ^{1i}_{\tau_n} I_{\{\tau_n<T\}} I_\Gamma.$$

The inequality (1.3.1) follows from here.

Note that for $\varepsilon_1, \varepsilon_2 \in]0, 1/4]$

$$\frac{(1+\varepsilon_1)(1+\varepsilon_2)}{1-\varepsilon_2} - 1 = \frac{\varepsilon_1 + 2\varepsilon_2 + \varepsilon_1\varepsilon_2}{1-\varepsilon_2} \le 4(\varepsilon_1+\varepsilon_2).$$

Since F is decreasing, we obtain from (1.3.1) that

$$F^i(\varepsilon_1)F^i(\varepsilon_2) \ge F^i(4(\varepsilon_1+\varepsilon_2))$$

for all $\varepsilon_1, \varepsilon_2 \in]0, 1/8]$. Using Lemma 1.3.5 below with $f = \ln F^i$, we get the needed bound. $\square$

Lemma 1.3.5 *Let $f :]0, x_0] \to \mathbf{R}$ be a decreasing function such that*

$$f(x_1) + f(x_2) \geq f(4(x_1 + x_2)), \qquad \forall\, x_1, x_2 \leq x_0. \tag{1.3.2}$$

Then there is $c > 0$ such that $f(x) \geq -cx^{1/3}$ for $x \in]0, x_0]$.

Proof. In the non-trivial case where $f(x_0) < 0$, the constant $\kappa = -\inf_{x \in]x_0/8, x_0]} f(x)/x$ is strictly greater than zero. Iterating the inequality $2f(x) \geq f(8x)$ we obtain that $2^n f(x) \geq f(2^{3n}x)$ for all $x \in]0, 2^{-3n}x_0]$ and all integers $n \geq 0$. Therefore,

$$\frac{f(x)}{x} \geq 2^{2n} \frac{f(2^{3n}x)}{2^{3n}x} = \frac{1}{4} x_0^{2/3} \left(\frac{2^{3(n+1)}}{x_0} \right)^{2/3} \frac{f(2^{3n}x)}{2^{3n}x}.$$

For $x \in]2^{-3(n+1)}x_0, 2^{-3n}x_0]$, the right-hand side dominates $-cx^{-2/3}$ with the constant $c := \kappa x_0^{2/3}/4$. Thus, the inequality $f(x)/x \geq -cx^{-2/3}$ holds on $]0, x_0]$. $\square$

1.4 Proof of Theorem 1.1.1

($\Leftarrow$) The arguments are standard. For any $\xi \in \phi_T A_0^{T\varepsilon}(T)$ and $Z \in \mathcal{M}_0^T(\widehat{K}^{\varepsilon *} \setminus \{0\})$ we have $EZ_T\xi \leq 0$ and this inequality is impossible for $\xi \in L^0(\mathbf{R}^d_+, \mathcal{F}_T)$, $\xi \neq 0$.

($\Rightarrow$) In view of Proposition 1.2.1 we need to consider the case where the universal chain is such that $P(\tau_n < T) > 0$ for every n and we can apply the results on functions F^i. Now the claim follows from the assertions below (cf. Prop. 3.7 and Th. 3.7 in [19]).

Proposition 1.4.1 *If $\sum_i F^i(\varepsilon) = 0$ for all $\varepsilon \in]0, 1]$, then the set $\mathcal{M}_0^T(\widehat{K}^{\varepsilon *} \setminus \{0\})$ is non-empty.*

Proof. Fix $\varepsilon \in]0, 1]$ and define a sequence of $\varepsilon_k \downarrow 0$, such that $\bar{\varepsilon}_N \uparrow \varepsilon$ where $\bar{\varepsilon}_1 = \varepsilon_1$,

$$\bar{\varepsilon}_N := (1 + \varepsilon_1) \prod_{k=2}^{N} \frac{1 + \varepsilon_k}{1 - \varepsilon_k} - 1, \qquad N \geq 2.$$

We extend arguments of the proof of Proposition 1.3.4 in the following way. Namely, we construct inductively an increasing sequence of integers $(n_N)_{N\ge 0}$ with $n_0 = 0$ and a sequence of $Z^{(N)} \in \mathcal{M}_0^{\tau_{n_N}}(\widehat{K}^{\bar\varepsilon_N *} \setminus \{0\})$ such that for $N = kd + r$ where $0 \le r \le d-1$

$$EZ^{(N)\,r+1}_{\tau_{n_N}} I_{\{\tau_{n_N} < T\}} \le 2^{-N}. \tag{1.4.3}$$

Since $F^1(\varepsilon) = 0$, Lemma 1.3.3 ensures the existence of $Z^1 \in \mathcal{M}^1(0, \varepsilon_1, n_1)$ with

$$EZ^{11}_{\tau_{n_1}} I_{\{\tau_{n_1} < T\}} \le 2^{-1}.$$

Put $Z^{(1)} := Z^1$. Take now $\delta_1 > 0$ such that

$$EZ^{(1)2}_{\tau_{n_1}} I_{\{\tau_{n_1} < T\}} I_A \le 2^{-3}$$

for every $A \in \mathcal{F}_{\tau_{n_1}}$ with $P(A) \le \delta_1$. Using again Lemma 1.3.3 (now for the second coordinate), we find an integer $n_2 > n_1$, a set $\Gamma_1 \in \mathcal{F}_{\tau_{n_1}}$ with $P(\Gamma_1) \le \delta_1 \wedge 2^{-3}$, and a process $Z^2 \in \mathcal{M}^2(\tau_{n_1}, \varepsilon_2, n_2)$ such that $Z^{22}_{\tau_{n_1}} = I_{\{\tau_{n_1} < T\}} / EI_{\{\tau_{n_1} < T\}}$ and

$$E(Z^{22}_{\tau_{n_2}} I_{\{\tau_{n_2} < T\}} | \mathcal{F}_{\tau_{n_1}}) \le \frac{I_{\{\tau_{n_1} < T\}}}{EI_{\{\tau_{n_1} < T\}}} [2^{-3} + I_{\Gamma_1}].$$

Put $Z^{(2)j}_t = Z^{(1)j}_t$ on $[0, \tau_{n_1}]$ and $Z^{(2)j}_t = Z^{2j}_t Z^{(1)j}_{\tau_{n_1}} / Z^{2j}_{\tau_{n_1}}$ on $]\tau_{n_1}, \tau_{n_2}]$, $j = 1, \dots, d$. Note that $Z^{(2)} \in \mathcal{M}_0^{\tau_{n_2}}(\phi^{-1}\mathrm{cone}\,\{\mathbf{1} + U_{\bar\varepsilon_2}\} \setminus \{0\})$ and

$$\begin{aligned} EZ^{(2)2}_{\tau_{n_2}} I_{\{\tau_{n_2} < T\}} &= P(\tau_{n_1} < T) EZ^{22}_{\tau_{n_2}} Z^{(1)2}_{\tau_{n_1}} I_{\{\tau_{n_2} < T\}} \\ &\le P(\tau_{n_1} < T) EZ^{(1)2}_{\tau_{n_1}} I_{\{\tau_{n_1} < T\}} E(Z^{22}_{\tau_{n_2}} I_{\{\tau_{n_2} < T\}} | \mathcal{F}_{\tau_{n_1}}) \le 2^{-2}. \end{aligned}$$

We continue this procedure passing at each step from the coordinate j to the coordinate $j+1$ for $j \le d-1$ and from the coordinate d to the first one.

Since $P(\tau_n = T) \uparrow 1$, there is a process Z such that $Z^{\tau_{n_N}} = Z^{(N)}$ for every N. The components of Z are strictly positive processes on $[0, T]$. The condition (1.4.3) ensures that they are martingales. Therefore, $Z \in \mathcal{M}_0^T(\widehat{K}^{\varepsilon *} \setminus \{0\})$. $\square$

Proposition 1.4.2 *Suppose that $\sum F^i \ne 0$. Then there is $\varepsilon \in]0,1]$ for which the property $NA^{w\varepsilon}$ (the notation is obvious) does not hold.*

Proof. At least one of functions, say, F^1, is not equal identically to zero. According to Proposition 1.3.4, we have the bound $F^1(\varepsilon) > e^{-c\varepsilon^{1/3}}$ for all sufficiently small ε. Hence, there is a stopping time σ dominated by certain τ_{n_0} on the set $\{\sigma < T\}$ such that

$$\inf_{Z\in\mathcal{M}^1(\sigma,\varepsilon,n)} EZ^1_{\tau_n} I_{\{\tau_n<T\}} > e^{-c\varepsilon^{1/3}}$$

for all sufficiently large n. Then for every $Z \in \mathcal{M}^1(\sigma,\varepsilon,n)$ we have that

$$E(Z^1_{\tau_n} I_{\{\tau_n=T\}}|\mathcal{F}_\sigma) \le 1 - e^{-c\varepsilon^{1/3}}.$$

Let us consider the sequence of random variables $\xi^n \in L^0(\mathbf{R}^d, \mathcal{F}_{\tau_n})$ such that the components $\xi^{n2} = \cdots = \xi^{nd} = 0$ and

$$\xi^{n1} = -I_{\{\sigma<T\}} + (1 - e^{-c\varepsilon^{1/3}})^{-1} I_{\{\sigma<T,\tau_n=T\}}.$$

Clearly,

$$E(Z_{\tau_n}\xi^n|\mathcal{F}_\sigma) \le -I_{\{\sigma<T\}} + (1 - e^{-c\varepsilon^{1/3}})^{-1} E(Z^1_{\tau_n} I_{\{\tau_n=T\}}|\mathcal{F}_\sigma) I_{\{\sigma<T\}} \le 0.$$

We have the inequality $EZ_{\tau_n}\xi^n \le 0$, and, therefore, by the superhedging theorem (see Th. 3.6.3 in [24]), ξ^n is the terminal value of an admissible process $\widehat{V} = \widehat{V}^B$ in the model having σ and τ_n as the initial and terminal dates, respectively. Note that on the non-null set $\{\sigma < T\}$ the limit of ξ^{n1} is strictly positive. To conclude we use the lemma below which one can get by applying, on each interval $[0,\tau_n]$, the Komlós-type result (Lemma 3.6.5 in [24], Lemma 3.5 in [23]) followed by the diagonal procedure. □

Lemma 1.4.3 *Suppose that* $\xi^n = \widehat{V}^n_{\tau_n}$ *where* $\widehat{V}^n + \mathbf{1} \in \widehat{K}^\varepsilon$ *and* $\xi^n \to \xi$ *a.s. as* $n \to \infty$. *Then there is a portfolio process* $\widehat{V}$ *such that* $\widehat{V} + \mathbf{1} \in \widehat{K}^\varepsilon$ *and* $\xi = \widehat{V}_T$.

Chapter 2

Financial Application

2.1 Comments on financial applications.

It is easily seen that for the case $d = 2$ our model is exactly the same as that of [19] and our theorem is Th. 1.1 therein. The only difference is that we use the "old-fashion" definition of the value processes. The reader is invited to verify that one can use the more sophisticated one as defined in [24] (following the original paper [6]) and get the same result. In the financial interpretation the cones K^ε and $\widehat{K}^\varepsilon$ are the solvency regions in the terms of the numéraire and physical units, respectively, V and $\widehat{V}$ are value processes, elements of $\mathcal{M}_0^T(\widehat{K}^{\varepsilon *}\setminus\{0\})$ are ε-consistent price systems, etc. The condition "$A_0^{T\varepsilon}(T) \cap L^0(\mathbf{R}^d_+, \mathcal{F}_T) = \{0\}$ for all ε" can be referred to as the *universal* NA^w*-property.*

In the case $d > 2$ the considered cones K^ε and $K^{\varepsilon *}$ do not correspond to a financial model (though sometimes the traditional terminology is still in use). What is important, our result can be applied to a wide class of financially meaningful models, even with varying transaction costs. To see this, let us consider the family of models of currency markets with the solvency cones given by the matrices of transaction costs coefficients $\Lambda^\varepsilon = (\lambda^\varepsilon_{ij})$ as follows:

$$K(\Lambda^\varepsilon) = \text{cone}\,\{(1+\lambda^\varepsilon_{ij})e_i - e_j,\ e_i,\ 1 \le i,j \le d\}.$$

Suppose that for every $\varepsilon \in]0,1]$ there is $\varepsilon' \in]0,1]$ such that $K(\Lambda^\varepsilon) \subseteq K^{\varepsilon'}$ and, vice versa, for any $\delta \in]0,1]$ there is $\delta' \in]0,1]$ such that $K^\delta \subseteq K(\Lambda^{\delta'})$. It is obvious that under this hypothesis Theorem 1.1.1 ensures that for the currency markets the $NA^w(\Lambda^\varepsilon)$-property holds for every $\varepsilon \in]0,1]$ if and only if an ε-consistent price system does exist for every $\varepsilon \in]0,1]$. The hypothesis is fulfilled if $\Lambda^\varepsilon \to 0$ and the duals $K^*(\Lambda^\varepsilon)$ have interiors containing $\mathbf{1}$, e.g., in the case where all $\lambda^\varepsilon_{ij} = \varepsilon$. Adding some extra arguments one can easily get the following corollary of the main theorem for the family of models with the efficient friction condition.

Proposition 2.1.1 *Suppose that* $\Lambda^\varepsilon \to 0$ *and* $\operatorname{int} K^*(\Lambda^\varepsilon) \neq \emptyset$ *for all* $\varepsilon \in]0,1]$. *Then*

$$NA^w(\Lambda^\varepsilon) \quad \forall \varepsilon \in]0,1] \quad \Leftrightarrow \quad \mathcal{M}_0^T(\widehat{K}^*(\Lambda^\varepsilon) \setminus \{0\}) \neq \emptyset \quad \forall \varepsilon \in]0,1].$$

Proof. ($\Rightarrow$) Let $\delta \in]0,1]$ and $w \in K^*(\Lambda^\delta)$. Then $w^i/w^j \leq 1 + \lambda^\delta_{ij} \to 1$ as $\delta \to 0$. It follows that $K^*(\Lambda^{\delta'}) \subseteq K^{\delta *}$ for some $\delta' \in]0,1]$. For the primary cones the inclusion is opposite. Thus, the assumed no-arbitrage property implies the no-arbitrage property in the formulation of Theorem 1.1.1. Take now $\varepsilon \in]0,1]$ and a vector $v \in \operatorname{int} K^*(\Lambda^\varepsilon) \cap U_1$. We define the operator

$$\psi_v : (x^1, ..., x^d) \mapsto (v^1 x^1, ..., v^d x^d).$$

Choose $\delta \in]0,1]$ such that $\psi_v(\mathbf{1} + U_\delta) \subset K^*(\Lambda^\varepsilon)$. By virtue of Theorem 1.1.1 there is $Z \in \mathcal{M}_0^T(\widehat{K}^{\delta *} \setminus \{0\})$. The process $\psi_v Z$ is a martingale. Since $\psi_v Z = \phi \psi_v \phi^{-1} Z$, it is an element of $\mathcal{M}_0^T(\widehat{K}^*(\Lambda^\varepsilon) \setminus \{0\})$.

For the proof of the reverse implication see the beginning of Section 1.4. $\square$

2.2 Richness of the Set of Consistent Price Systems

The following condition of "richness" of consistent price systems plays an important role in the continuous-time theory of financial markets with transaction costs.

$\mathbf{B}^\varepsilon$ Let $\xi \in L^0(\mathbf{R}^d, \mathcal{F}_t)$. If $Z_t\xi \geq 0$ for all $Z \in \mathcal{M}_0^T(\widehat{K}^{\varepsilon *} \setminus \{0\})$, then $\xi \in \widehat{K}_t^\varepsilon$ (a.s.).

Simple argument (see, e.g., [24], 3.6.3) shows that $\mathbf{B}^\varepsilon$ is fulfilled for the model with constant transaction costs if S admits an equivalent martingale measure. Its minor changes leads to the next result which seems to be useful interesting in the setting of families of models with vanishing transaction costs:

Proposition 2.2.1 *Suppose that* $\mathcal{M}_0^T(\widehat{K}^{\varepsilon *} \setminus \{0\}) \neq \emptyset$ *for all* $\varepsilon \in]0,1]$. *Then the condition* $\mathbf{B}^\varepsilon$ *holds for all* $\varepsilon \in]0,1]$.

Proof. Take $w \in \operatorname{int} K^{\varepsilon *}$ with $|w| = 1$. For all sufficiently small $\delta > 0$ we have the inclusion $w + U_\delta \subset K^{\varepsilon *}$. Take $Z \in \mathcal{M}_0^T(\widehat{K}^{\delta *} \setminus \{0\})$ and consider the martingale $Z^w = (w^1 Z^1, \ldots, w^d Z^d)$. Note that $\phi_t Z_t = \rho_t \tilde{Z}_t$ where $\rho_t > 0$ and $\tilde{Z}_t \in \mathbf{1} + U_\delta$. Then $\phi_t Z_t^w = \rho_t \tilde{w}_t$ where $\tilde{w}_t^i = w^i \tilde{Z}_t^i$. According to our definition, $\tilde{w}_t$ takes values in $w + U_\delta \subset K^{\varepsilon *}$. Therefore, $Z^w \in \mathcal{M}_0^T(\widehat{K}^{\varepsilon *} \setminus \{0\})$ and $Z^w \xi \geq 0$. The inequality implies that $\tilde{w}_t \eta_t \geq 0$ where $\eta_t(\omega) = \phi_t^{-1}(\omega)\xi(\omega)$. Letting $\delta \to 0$, we obtain that also $w\eta_t \geq 0$. The latter inequality holds for all $w \in K^{\varepsilon *}$. Hence, $\phi_t^{-1}\xi \in K^\varepsilon$ and $\xi \in \widehat{K}_t^\varepsilon$. $\Box$

Part II

Limit Behavior of Option Hedging Sets under Transaction Costs

Though continuous trading is a part of the standard paradigm of modern finance, in practice, usually, portfolio revisions are done along a discrete–time greed. In the case of proportional transaction costs the agents know the order of total number of transactions and agree between them on a transaction costs coefficient: for more frequent revisions one can expect a smaller level of the latter. It is well-known that the straightforward discrete-time approximation for the option price (suitably defined) may not lead to a "theoretical" price generated by the continuous-time model. One of the remedy is to modify the model as was suggested in the pioneering work by Leland [31] and studied afterwords by a number of authors (see the book [24] and references therein and also more recent papers [8], [32]). In the Leland–Lott model the transaction costs are decreasing with the rate $n^{-1/2}$. The terminal values of portfolios approximate the pay-off of the option and the limit of their initial values is declared to be a fair option price accepted by practitioners as realistic.

In [30] Kusuoka considered a sequence of discrete-time two-asset models where the transaction costs are also decreasing with the rate $n^{-1/2}$. He calculated the limit of super-replication prices which happens to be different from that of the limiting continuous-time model based on a geometric Brownian motion.

The aim of this paper is to place the Kusuoka approach in the now standard geometric formalism of the theory of markets with transaction costs as presented in [24]. The main idea of the theory is to consider all objects as vector-valued: initial endowments, portfolios, contingent claims etc. and appeal to "physical units" in conjunction with quotes in terms of the numéraire. "Vectorization" of the theory fills the gap between the approach of classical mathematical finance (where everything is expressed in money) and that of mathematical economics (where the vectors of commodities can be considered as the primary objects). Accordingly, the initial endowments which allows the investor to run a self-financing portfolio to super-replicate a contingent claim is a subset of $\mathbb{R}^{1+d}$ where d is a number of risky assets. In this mainstream, the contingent claim of interest is a quantity of physical units.

The hedging theorem, which is a fundamental result of the theory, gives a dual description of this set in terms of the so-called consistent price systems, i.e. martingales evolving in the dual to the solvency cones in physical units. A sequence of models generates the sets of hedging endowments and the aim is to find a limit for a sequence of these sets.

In the following chapter, we focus on models of "stock" market where it is assumed that all transactions pass through the money. Consequently, we consider a rather specific sequence of simple polyhedral conic models given by transaction costs penalizing direct transactions between assets. In the case $d = 1$, it is essentially the same as that of Kusuoka. The minor difference is in the use, to express the price processes, of the "stochastic" exponential instead of classical one. We prove that the sequence of sets Γ^n of hedging endowments converges to a limit in the sense of closed topology and we describe the limit. This makes clearly the difference between our result and that of [30]: Kusuoka considered the limiting behavior of x_n where $(x_n, 0) \in \mathbb{R}^2$ are the points laying in the intersection of the boundary of Γ^n with the axis of abscissae (that is, corresponding to the minimal initial endowments in money and zero in stock), while we study the limiting behavior of the whole sets. In the multidimensional setting, for a sequence of models given by a general matrix with transaction costs coefficients of the form $n^{-1/2}\Lambda$, our theorem combined with dominance considerations gives bounds for $\operatorname{Li}\Gamma^n$ and $\operatorname{Ls}\Gamma^n$, the topological $\liminf \Gamma^n$ and $\limsup \Gamma^n$. The precise limiting behavior of Γ^n in this case remains an open problem.

Cette partie est issue des articles [15, 16].

Chapter 3

The Multidimensional Mainstream

3.1 Model and main result

We consider a sequence of models of stock market with traded numéraire (”money”) and d stocks. All the orders are ”buy ith stock” or ”sell ith stock”, that is the transactions pass through money. The operations on the ith stock are charged with the same proportional transaction cost coefficients λ^{ni}. Namely, increasing the value of the ith position in one unit of the numéraire leads to diminishing in $1+\lambda^{ni}$ the money account while decreasing the ith position in $1+\lambda^{ni}$ unit of the numéraire increases the money account in one unit. We fix as transaction cost parameter the d-dimensional vector $\lambda \in]0,\infty[^d$ and the sequence

$$\lambda^n = \lambda\sqrt{T/n}.$$

Price processes

We define in this subsection continuous–time models whose price processes are piecewise constant on the intervals forming uniform partitions of $[0,T]$. Of course, these models are in one-to-one correspondence with discrete–time models. Fix, as the basic parameters, vectors $\mu \in \mathbb{R}^d$, $\sigma \in]0,\infty[^d$ and put, for $n \geq 1$,

$$\mu^n = \mu T/n, \qquad \sigma^n = \sigma\sqrt{T/n}.$$

We consider, on some probability space $(\Omega, \mathcal{F}, P)$, a double indexed family of i.i.d. random variables $\{\xi_k^i;\ k \leq n,\ 1 \leq i \leq d\}$, where ξ_k^i take values in $\{-1, 1\}$ and $P(\xi_k^i = 1) = 1/2$. Put

$$t_k = t_k^n := kT/n.$$

Define the process $S_t^n = (S_t^{n0}, S_t^{n1}, \cdots, S_t^{nd})$ where $S_0^n = \mathbf{1} := (1, \cdots, 1)$, $S_t^{n0} = 1$ and, for $i \geq 1$,

$$S_t^{ni} = \prod_{m=1}^{k} \left(1 + \mu^{ni} + \sigma^{ni}\xi_m^i\right), \qquad t \in [t_k, t_{k+1}[, \quad 0 \leq k \leq n-1, \tag{3.1.1}$$

for sufficiently large n. We associate with this process its natural filtration $\mathbf{F^n} = (\mathcal{F}_t^n)$ where $\mathcal{F}_t^n := \sigma\{S_r^n,\ r \leq t\}$. In this setting the stochastic basis $(\Omega, \mathcal{F}, \mathbf{F}^n, P)$ together with the process S^n models the price evolution of one non-risky and d risky assets, the latter measured in the non-risky one serving as the numéraire.

Transaction costs

The solvency region is defined by the cone

$$K^n = \text{cone}\left\{\big(1 + \lambda^{ni}\big)e^i - e^0,\ \big(1 + \lambda^{ni}\big)e^0 - e^i,\ 1 \leq i \leq d\right\},$$

where, consistently with current notations, e^0 is the first canonical vector of $\mathbb{R}^{1+d}$. Its (positive) dual cone is

$$K^{n*} = \left\{w \in \mathbb{R}^{1+d} : \frac{1}{1+\lambda^{ni}} \leq \frac{w^i}{w^0} \leq 1 + \lambda^{ni},\ 1 \leq i \leq d\right\}.$$

The dynamics of the portfolio value is given the $(d+1)$-dimensional piecewise constant process V defined as the solution of linear controlled stochastic equation

$$V_0 = v \in K^n, \quad dV_t^i = V_{t-}^i dS_t^{ni}/S_{t-}^{ni} + dB_t^i, \qquad 0 \leq i \leq d,$$

where the components of the control B (the strategy of portfolio revisions) are

$$B^i = \sum_{k=1}^{n} B^i_k \mathbb{I}_{]t_{k-1},t_k]},$$

where B^i_k is $\mathcal{F}^n_{t_{k-1}}$-measurable and $\Delta B_{t_k} = B_{t_k} - B_{t_{k-1}} \in L^0(-K^n, \mathcal{F}^n_{t_{k-1}})$. The set of such processes V with initial value v is denoted by $\mathcal{A}^n_v$ while the notation $\mathcal{A}^n_v(T)$ is reserved for the set of their terminal values V_T.

Using the diagonal random operator

$$\phi^n_t : (x^0, x^1, \cdots, x^d) \mapsto (x^0, x^1/S^{n1}_t, \cdots, x^d/S^{nd}_t)$$

define the random cone $\widehat{K}^n_t = \phi^n_t K^n$ (describing the solvency region in terms of physical units) with the dual $\widehat{K}^{n*}_t = (\phi^n_t)^{-1} K^{n*}$ which can be represented in a more explicit way as

$$\widehat{K}^{n*}_t = \left\{ w \in \mathbb{R}^{1+d} : \frac{1}{1+\lambda^{ni}} S^{ni}_t \leq \frac{w^i}{w^0} \leq (1+\lambda^{ni}) S^{ni}_t,\ 1 \leq i \leq d \right\}. \quad (3.1.2)$$

Hedging sets

Our aim is to price a European option which pay-off expressed in term of physical units is of the form $F(S^n)$. The function $F : \mathbb{D}(\mathbb{R}^{1+d}) \to \mathbb{R}^{1+d}_+$ is supposed to be bounded and continuous in the Skorohod topology on $\mathbb{D}(\mathbb{R}^{1+d})$. Let Γ^n be the set of initial endowments from which one can start a self-financing portfolio process with the terminal value dominating the contingent claim $F(S^n)$, i.e.

$$\Gamma^n = \{ v \in \mathbb{R}^{1+d} :\ (\phi^n_T)^{-1} F(S^n) \in \mathcal{A}^n_v(T) \}.$$

Let $\mathcal{Q} = \mathcal{Q}^\lambda$ be the set of probability measures Q on $\{1\} \times \mathbb{C}(\mathbb{R}^d)$ (endowed with the Borel σ-algebra) which are the distributions of the continuous martingales $U_t = (1, U^1_t, \cdots, U^d_t)$, $t \in [0,T]$ such that

$$U^i = \mathcal{E}(L^i) = e^{L - \frac{1}{2}\langle L^i \rangle},$$

where L^i are square integrable continuous martingales with the absolute continuous characteristics satisfying

$$\max\{\sigma^i(\sigma^i - 2\lambda^i), 0\} \le \frac{d\langle L^i\rangle}{dt} \le \sigma^i(\sigma^i + 2\lambda^i), \tag{3.1.3}$$

$$\langle L^i, L^j\rangle = 0, \qquad i \ne j. \tag{3.1.4}$$

We send the reader to [22], Chapter 1 for more detailed information on quadratic characteristics and notations.

Remark 3.1.1 *Without loss of generality we may assume that the processes L^i are stochastic integrals with respect to a d-dimensional Brownian motion. Indeed, according to [28], Theorem 3.4.2, there is a filtered probability space $(\Omega, \mathcal{F}, \mathbf{F}, R)$ with a Brownian motion B and the matrix-valued process g such that R-a.s. we have $L^i = g^i \cdot B$ with*

$$\langle L^i, L^j\rangle_t = \int_0^t (gg')^{ij} ds.$$

We put

$$\Gamma := \Gamma(\lambda) := \Big\{ v \in \mathbb{R}^{1+d} : \sup_{Q\in\mathcal{Q}^\lambda} E^Q\left(w_T F(w) - \mathbf{1}v\right) \le 0 \Big\}.$$

The reference to λ will be omitted when there is no ambiguity.

Convergence of sets and main result

Recall basic definitions concerning the topology of closed convergence on the space of closed subsets of $\mathbb{R}^{1+d}$, see, e.g., [1], [20].

Let E^n be a sequence of subsets of $\mathbb{R}^{1+d}$. Then:

(i) A point $v \in \mathbb{R}^{1+d}$ belongs to the **topological lim sup**, denoted Ls E_n, if for every neighborhood V of v there are infinitely many n with $V \cap E^n \ne \emptyset$.

(ii) A point $v \in \mathbb{R}^{1+d}$ belongs to the **topological lim inf**, denoted Li E_n, if for every neighborhood V of v, we have $V \cap E^n \ne \emptyset$ for all but finitely many n.

(iii) If Ls $E_n =$ Li $E_n = E$, then the set E is called the **closed limit** of the sequence E^n.

The set E is the closed limit of E^n if the following properties hold:

(i) For any $v \in E$, there exists a sequence of vectors $v^n \in E^n$, such that $v^n \to v$.

(ii) For any convergent subsequence of a sequence of vectors $v^n \in E^n$, the limit v belongs to E.

The main result of the paper is the following statement.

Theorem 3.1.2 *The set Γ is the closed limit of the sequence of the sets Γ^n.*

3.2 Hedging theorem and weak convergence

We denote by $\mathcal{M}^n$ the set of normalized consistent price systems for the nth model, i.e. of the $\mathbf{F}^n$-martingales Z such that $Z_t \in \widehat{K}_t^{n*} \setminus \{0\}$ and $Z_0^0 = 1$. According to [24], Chapter 3,

$$\Gamma^n = \left\{ v \in \mathbb{R}^{1+d} : \; vZ_0 \geq EZ_T F(S^n) \; \text{ for all } Z \in \mathcal{M}^n \right\}. \tag{3.2.5}$$

This identity is the so-called hedging theorem claiming that one can super-replicate the contingent claim if and only if the value of the initial endowments is not less than the expectation of the value of the contingent claim whatever a consistent price system is used to the comparison. The theorem holds under the assumption of the existence of a strictly consistent price system, fulfilled for our models.

We obtain our convergence result for Γ^n by using the representation (3.2.5) and the theory of weak convergence of measures.

Tightness

Let us consider a sequence $Z^n \in \mathcal{M}^n$. The strictly positive martingale Z^{n0} is the density process of the probability measure $Q^n = Z_T^{n0} P$ and the components of the processes $M^n := Z^n/Z^{n0}$ are strictly positive Q^n-martingales with respect to the filtration $\mathbf{F}^n$. We show that the sequence M^n is Q^n-tight or, more precisely, that the sequence of laws $\mathscr{L}(M^n|Q^n)$ is tight.

To simplify formulae we use for the averaging with respect to Q^n the symbol E^n instead of E^{Q^n}.

In view of (3.1.2)

$$\frac{1}{1+\lambda^{ni}} S^{ni} \le M^{ni} \le (1+\lambda^{ni}) S^{ni}, \qquad 1 \le i \le d. \tag{3.2.6}$$

Let us define the piecewise constant processes L^n ("stochastic logarithms" of M^n)

$$L^{ni} := (M^{ni}_-)^{-1} \cdot M^{ni},$$

which jumps only at the points t_k, $k \ge 1$. Namely, we have:

$$\begin{aligned} \Delta L^{ni}_{t_k} &= (M^{ni}_{t_{k-1}})^{-1} \Delta M^{ni}_{t_k} = (M^{ni}_{t_{k-1}})^{-1} (M^{ni}_{t_k} - M^{ni}_{t_{k-1}}) \\ &= \exp(\Delta \ln M^{ni}_{t_k}) - 1. \end{aligned} \tag{3.2.7}$$

Lemma 3.2.1 *We have the following asymptotics:*

$$||\Delta \ln M^n||_T = O(n^{-1/2}), \tag{3.2.8}$$
$$||\Delta L^n||_T = O(n^{-1/2}), \tag{3.2.9}$$
$$||\Delta \ln M^n - \Delta L^n||_T = O(n^{-1}), \tag{3.2.10}$$
$$\sup_{k \le n} \left| E^n[\Delta \ln M^n_{t_k} | \mathcal{F}^n_{t_{k-1}}] \right| = O(n^{-1}). \tag{3.2.11}$$

Proof. Directly from the definition (3.1.1) of the process S^n we have the bounds

$$\ln\left(1+\mu^{ni}-\sigma^{ni}\right) \le \Delta \ln S^{ni}_{t_k} \le \ln\left(1+\mu^{ni}+\sigma^{ni}\right), \qquad i \ge 1,$$

allowing us to derive from (3.2.6) the inequalities

$$-2\ln\left(1+\lambda^{ni}\right) + \ln\left(1+\mu^{ni}-\sigma^{ni}\right) \le \Delta \ln M^{ni}_{t_k} \le 2\ln\left(1+\lambda^{ni}\right) + \ln\left(1+\mu^{ni}+\sigma^{ni}\right),$$

implying (3.2.8) in virtue of the assumed asymptotics for coefficients. The relation (3.2.9) follows from (3.2.7) and (3.2.8).

Since

$$\Phi_1(z) := \ln(1+z) - z = O(z^2), \qquad z \to 0,$$

the relation (3.2.9) implies that

$$||\Phi_1(\Delta L^{ni})||_T = O(n^{-1}).$$

Noting that

$$\Delta \ln M^{ni}_{t_k} = \Delta L^{ni}_{t_k} + \Phi_1\big(\Delta L^{ni}_{t_k}\big),$$

and taking into account that

$$E^n[\Delta L^{ni}_{t_k}|\mathcal{F}^n_{t_{k-1}}] = 0,$$

we obtain (3.2.10) and (3.2.11). □

Lemma 3.2.2 *Let $m \geq 1$ be an integer. Then*

$$\sup_n E^n||M^n||_T^{2m} < \infty, \qquad \sup_n E^n|| \ln M^n||_T^{2m} < \infty. \tag{3.2.12}$$

There exists a constant κ such that for any n and $l \leq n$ we have the bound

$$E^n \sup_{k \leq n-l} \big|\langle M^{ni}\rangle_{t_{k+l}} - \langle M^{ni}\rangle_{t_k}\big|^2 \leq \kappa(l/n)^2 = \kappa T^2(t_{k+l}-t_k)^2. \tag{3.2.13}$$

Proof. Using (3.2.7), the binomial formula, the martingale property of L^{ni}, and the estimate (3.2.9) we have:

$$\begin{aligned} E^n(M^{ni}_{t_k})^{2m} &= E^n(M^{ni}_{t_{k-1}})^{2m}(1+\Delta L^{ni}_{t_k})^{2m} \\ &= E^n(M^{ni}_{t_{k-1}})^{2m}\left(1+\sum_{j=1}^{2m}\binom{2m}{j}(\Delta L^{ni}_{t_k})^j\right) \\ &= E^n(M^{ni}_{t_{k-1}})^{2m}\left(1+\sum_{j=2}^{2m}\binom{2m}{j}(\Delta L^{ni}_{t_k})^j\right) \\ &\leq E^n|M^{ni}_{t_{k-1}}|^{2m}(1+c_m n^{-1})^{2m}, \end{aligned}$$

for some constant $c_m > 0$. It follows that

$$E^n(M_T^{ni})^{2m} \le (1 + c_m n^{-1})^{2mn} \le \text{const},$$

and the Doob inequality implies that

$$\sup_n E^n ||M^n||_T^{2m} < \infty.$$

Put $X_{t_k}^n := \ln M_{t_k}^n - E^n[\ln M_{t_k}^n | \mathcal{F}_{t_{k-1}}^n]$. By (3.2.11)

$$||\Delta \ln M^n - X^n||_T = \sup_{k \le n} \left| E^n[\Delta \ln M_{t_k}^n | \mathcal{F}_{t_{k-1}}^n] \right| = O(n^{-1}). \tag{3.2.14}$$

Combining this with (3.2.8) we obtain that

$$||X^n||_T = O(n^{-1/2}). \tag{3.2.15}$$

By the Burkholder–Davis–Gundy inequality

$$E^n \left| \sum_{j \le k} X_{t_j}^n \right|^{2m} \le C_m E^n \left(\sum_{j \le n} X_{t_j}^{n2} \right)^m, \qquad k \le n,$$

and the claim (3.2.12) follows from (3.2.15).

Let $k \in [0, n]$. For any $l \le n-k$ we get, using the relation (3.2.9) which provides us a deterministic bound for $||\Delta L^n||_T$, that

$$\begin{aligned}
\langle M^{ni} \rangle_{t_{k+l}} - \langle M^{ni} \rangle_{t_k} &= \sum_{i=1}^{l} E^n \big((\Delta M_{t_{k+i}}^{ni})^2 | \mathcal{F}_{t_{k+i-1}} \big) \\
&= \sum_{i=1}^{l} E^n \big((M_{t_{k+i-1}}^{ni})^2 (\Delta L_{t_{k+i}}^{ni})^2 | \mathcal{F}_{t_{k+i-1}} \big) \\
&\le cln^{-1} ||M^{ni}||_T^2,
\end{aligned}$$

where c is a constant. The inequality (3.2.13) now follows obviously from this estimate because by virtue of (3.2.12) the sequence $E^n||M^{ni}||_T^4$ is bounded by a constant. $\square$

For a function $\alpha \in \mathbb{D}(\mathbb{R})$ we define the modulus of continuity $w(\alpha, \delta)$, $\delta > 0$, by the formula

$$w(\alpha, \delta) := \sup\{|\alpha_{t+h} - \alpha_t| : \ t \in [0, T - \delta], \ h \in [0, \delta]\}.$$

The inequality (3.2.13) implies the following estimate:

Corollary 3.2.3 *There is a constant $\kappa > 0$ such that for for any $\delta > 0$ we have, for all sufficiently large n, the inequality*

$$E^n \big| w(\langle M^{ni} \rangle, \delta) \big|^2 \le \kappa(\delta + T/n)^2. \tag{3.2.16}$$

Lemma 3.2.4 *For $i \ne j$*

$$\sup_{k \le n} \Big| E^n[\Delta L^{ni}_{t_k} \Delta L^{nj}_{t_k} | \mathcal{F}^n_{t_{k-1}}] \Big| = O(n^{-3/2}).$$

Proof. Note that

$$E^n[\Delta L^{ni}_{t_k} \Delta L^{nj}_{t_k} | \mathcal{F}^n_{t_{k-1}}] = E^n[(\Delta L^{ni}_{t_k} - \Delta \ln M^{ni}_{t_k}) \Delta L^{nj}_{t_k} | \mathcal{F}^n_{t_{k-1}}] \\ + E^n[\Delta \ln M^{ni}_{t_k} \Delta L^{nj}_{t_k} | \mathcal{F}^n_{t_{k-1}}].$$

Using first the estimate (3.2.9) and then (3.2.10) and (3.2.11) we get the result. □

Lemma 3.2.5 *For $k \le l \le n$ we have the following inequalities :*

$$\begin{aligned} -2E^n[\ln M^{ni}_{t_l} - \ln M^{ni}_{t_k} | \mathcal{F}^n_{t_k}] &\le (t_l - t_k)\sigma^i(\sigma^i + 2\lambda^i) + \lambda^{i2} T n^{-1} + (t_l - t_k) R_n, \\ -2E^n[\ln M^{ni}_{t_l} - \ln M^{ni}_{t_k} | \mathcal{F}^n_{t_k}] &\ge (t_l - t_k)\sigma^i(\sigma^i - 2\lambda^i) - \lambda^{i2} T n^{-1} - (t_l - t_k) R_n, \end{aligned}$$

where $R_n = O(n^{-1/2})$ does not depend on k and l.

Since $\ln M$ is a Q^n-supermartingale we have also that

$$-2E^n[\ln M^{ni}_{t_l} - \ln M^{ni}_{t_k} | \mathcal{F}^n_{t_k}] \ge 0.$$

Proof. Fix $i \ne 0$. Combining (3.2.10) and (3.2.11), we get that

$$\sup_{j \le n} \Big| \ln M^{ni}_{t_j} - E^n[\ln M^{ni}_{t_j} | \mathcal{F}^n_{t_{j-1}}] - \Delta L^{ni}_{t_j} \Big| = O(n^{-1}). \tag{3.2.17}$$

Put

$$\Phi_2(z) := \ln(1+z) - z + z^2/2 = O(z^3), \qquad z \to 0.$$

According to (3.2.9),

$$||\Phi_2(\Delta L^{ni})||_T = O(n^{-3/2}). \tag{3.2.18}$$

Recalling $\ln(1 + \Delta L^{ni}) = \Delta \ln M^{ni}$ it is easy to check the identity

$$\begin{aligned}
&2\Delta \ln M^{ni}_{t_j} - 2\Delta L^{ni}_{t_j} + (\ln M^{ni}_{t_j} - E^n[\ln M^{ni}_{t_j}|\mathcal{F}^n_{t_{j-1}}])^2 \\
&\quad = 2\Phi_2(\Delta L^{ni}_{t_j}) + (\ln M^{ni}_{t_j} - E^n[\ln M^{ni}_{t_j}|\mathcal{F}^n_{t_{j-1}}] - \Delta L^{ni}_{t_j})^2 \\
&\qquad + 2\Delta L^{ni}_{t_j}(\ln M^{ni}_{t_j} - E^n[\ln M^{ni}_{t_j}|\mathcal{F}^n_{t_{j-1}}] - \Delta L^{ni}_{t_j}).
\end{aligned}$$

Using (3.2.18), (3.2.17), and (3.2.9) we obtain that

$$\sup_{j\le n}\Big|E^n[2(\Delta \ln M^{ni}_{t_j}) + (\ln M^{ni}_{t_j} - E^n[\ln M^{ni}_{t_j}|\mathcal{F}^n_{t_{j-1}}])^2|\mathcal{F}^n_{t_{j-1}}]\Big| = O(n^{-3/2}).$$

This relation combined with (3.2.8) and (3.2.11) gives us the following bound

$$\sup_{j\le n}\Big|E^n[2(\Delta \ln M^{ni}_{t_j}) + (\Delta \ln M^{ni}_{t_j})^2|\mathcal{F}^n_{t_{j-1}}]\Big| = O(n^{-3/2}). \tag{3.2.19}$$

Putting $Y^{ni} := \ln M^{ni} - \ln S^{ni}$ and using (3.2.6), we get that

$$||Y^{ni}||_T \le \ln(1 + \lambda^{ni}) \le \lambda^{ni}, \tag{3.2.20}$$

and

$$\begin{aligned}
\Delta \ln M^{ni}_{t_j} + Y^{ni}_{t_{j-1}} &= \ln M^{ni}_{t_j} - \ln S^{ni}_{t_{j-1}} \\
&= Y^{ni}_{t_j} + \ln\big(1 + \mu^{ni} + \sigma^{ni}\xi^i_j\big).
\end{aligned}$$

With these observations we obtain the bound

$$\begin{aligned}
\sup_{j\le n}\Big|&E^n[(\Delta \ln M^{ni}_{t_j} + Y^{ni}_{t_{j-1}})^2 - (Y^{ni}_{t_j})^2|\mathcal{F}^n_{t_{j-1}}] \\
&-(\sigma^{ni})^2 - 2\sigma^{ni}E^n[Y^{ni}_{t_j}\xi^i_j|\mathcal{F}^n_{t_{j-1}}]\Big| = O(n^{-3/2}).
\end{aligned} \tag{3.2.21}$$

Having the identity

$$\begin{aligned}
&2\Delta \ln M^{ni}_{t_j} + \Delta(Y^{ni}_{t_j})^2 \\
&\quad = [2\Delta \ln M^{ni}_{t_j} + (\Delta \ln M^{ni}_{t_j})^2] - [(\Delta \ln M^{ni}_{t_j} + Y^{ni}_{t_{j-1}})^2 - (Y^{ni}_{t_j})^2] \\
&\qquad + 2Y^{ni}_{t_{j-1}}\Delta \ln M^{ni}_{t_j},
\end{aligned}$$

we deduce from (3.2.11), (3.2.19), (3.2.20) and (3.2.21) the relation

$$\begin{aligned}
\sup_{j\le n}\Big|&E^n[2\Delta \ln M^{ni}_{t_j} + \Delta(Y^{ni}_{t_j})^2|\mathcal{F}^n_{t_{j-1}}] \\
&+(\sigma^{ni})^2 + 2\sigma^{ni}E^n[Y^{n}_{t_j}\xi^i_j|\mathcal{F}^n_{t_{j-1}}]\Big| = O(n^{-3/2}).
\end{aligned}$$

Taking into account the bounds

$$|Y^{ni}_{t_j}\xi^i_j| \le \lambda^{ni}, \qquad \Delta|(Y^{ni}_{t_j})^2| \le (\lambda^{ni})^2 = (\lambda^i)^2 T n^{-1},$$

we obtain, for some constant $\kappa > 0$ which does not depend on k, that

$$\begin{aligned}
-2E^n[\Delta \ln M^{ni}_{t_j}|\mathcal{F}^n_{t_{j-1}}] &\le (\sigma^{ni})^2 + 2\sigma^{ni}\lambda^{ni} + E^n[\Delta(Y^{ni}_{t_j})^2|\mathcal{F}^n_{t_{j-1}}] + \kappa n^{-3/2},\\
-2E^n[\Delta \ln M^{ni}_{t_j}|\mathcal{F}^n_{t_{j-1}}] &\ge (\sigma^{ni})^2 - 2\sigma^{ni}\lambda^{ni} + E^n[\Delta(Y^{ni}_{t_j})^2|\mathcal{F}^n_{t_{j-1}}] - \kappa n^{-3/2}.
\end{aligned}$$

It follows that for any $m \le n-k$ we have the inequalities

$$\begin{aligned}
-2E^n[\ln M^{ni}_{t_{k+m}} - \ln M^{ni}_{t_k}|\mathcal{F}^n_{t_k}] &\le mn^{-1}T\sigma^i(\sigma^i + 2\lambda^i)\\
&\quad + E^n[(Y^{ni}_{t_{k+m}})^2 - (Y^{ni}_{t_k})^2|\mathcal{F}^n_{t_k}] + \kappa m n^{-3/2},\\
-2E^n[\ln M^{ni}_{t_{k+m}} - \ln M^{ni}_{t_k}|\mathcal{F}^n_{t_k}] &\ge mn^{-1}T\sigma^i(\sigma^i - 2\lambda^i)\\
&\quad + E^n[(Y^{ni}_{t_{k+m}})^2 - (Y^{ni}_{t_k})^2|\mathcal{F}^n_{t_k}] - \kappa m n^{-3/2}
\end{aligned}$$

implying the claim of the lemma with $R_n = \kappa T^{-1} n^{-1/2}$. $\square$

The next lemma concludes this technical section with a tightness result on sequences of martingales from $\mathcal{M}^n$.

Lemma 3.2.6 *Let $Z^n \in \mathcal{M}^n$, $M^n := Z^n/Z^{n0}$, and $Q^n := Z^{n0}_T P$. The sequence of probability measures $\tilde{Q}^n := \mathscr{L}(M^n|Q^n)$ is tight and each limit point belongs to $\mathcal{Q}$.*

Proof. We shall use the same terminology as in [22] and write "a sequence M^n is Q^n-tight" etc. with understanding that this is a statement concerning the laws $\tilde{Q}^n$.

We apply Th. VI.4.13 and Prop. VI.3.26 of [22] to prove that the sequence M^n is Q^n-C-tight. Indeed, the sequence of initial values M^n_0 is bounded, the sequence of processes $\sum_i \langle M^{ni}\rangle$ is C-tight by virtue of Th. 15.5 in [3] (its assumption is ensured by Corollary 3.2.3) and

$$Q^n\left(||\Delta M^n||_T > \delta\right) \le \frac{1}{\delta^2} E^n\, ||\Delta M^n||^2_T \le \frac{1}{\delta^2} E^n ||\Delta L^n||^2_T ||M^n||^2_T \to 0, \quad n \to 0,$$

by virtue of (3.2.9) and (3.2.12). Thus, the assumptions of the indicated references are verified.

Take an arbitrary limit point Q. By above, it charges only $\{1\} \times \mathbb{C}(\mathbb{R}^d)$. Abusing the notation, we write that Q is a weak limit of the whole sequence $\tilde{Q}^n = \mathscr{L}(M^n|Q^n)$. By virtue of (3.2.12) the sequence of random variables $||M^n||_T$ is uniformly integrable (with respect to Q^n) and, therefore, the coordinate process $w = (w_t)_{t\in[0,T]}$ is a Q-martingale with respect to the natural filtration, see [22], IX.1. It remains to check (3.1.3) and (3.1.4).

Fix $s < t \le T$. Obviously with Lemma 3.2.5, we have

$$-2E^n\big[\ln M_t^{ni} - \ln M_s^{ni} + \frac{1}{2}\sigma^i\big(\sigma^i + 2\lambda^i\big)(t-s)\big|\mathcal{F}_s^n\big] \le \frac{\kappa}{\sqrt{n}},$$

$$-2E^n\big[\ln M_t^{ni} - \ln M_s^{ni} + \frac{1}{2}\sigma^i\big(\sigma^i - 2\lambda^i\big)(t-s)\big|\mathcal{F}_s^n\big] \ge \frac{\kappa}{\sqrt{n}},$$

with the constant $\kappa > 0$ which does not depend on t and s. Let the function $g : [0,T] \times \mathbb{D}(\mathbb{R}^{1+d}) \to \mathbb{R}_+$ be bounded continuous in the product of the usual topology on $[0,T]$ and the Skorohod topology on $\mathbb{D}(\mathbb{R}^{1+d})$ and non-anticipating, i.e. $w \mapsto g_t(w)$ is $\sigma\{w_s, s \le t\}$-measurable for any t. By virtue of (3.2.12), the uniform integrability of the sequence $||\ln M^n||_T$, we have:

$$\lim_{n\to\infty} \; -2E^{\tilde{Q}^n} g_s(w)\big(\ln w_t^i - \ln w_s^i + \frac{1}{2}\sigma^i\big(\sigma^i + 2\lambda^i\big)(t-s)\big) \le 0,$$

and

$$\lim_{n\to\infty} \; -2E^{\tilde{Q}^n} g_s(w)\big(\ln w_t^i - \ln w_s^i + \frac{1}{2}\sigma^i\big(\sigma^i - 2\lambda^i\big)(t-s)\big) \ge 0.$$

This leads to the bounds

$$-2E^Q g_s(w)\big(\ln w_t^i - \ln w_s^i\big) \le E^Q g_s(w)\sigma^i\big(\sigma^i + 2\lambda^i\big)(t-s),$$

and

$$-2E^Q g_s(w)\big(\ln w_t^i - \ln w_s^i\big) \ge E^Q g_s(w)\sigma^i\big(\sigma^i - 2\lambda^i\big)(t-s).$$

Since $(w_t^i)_{t\in[0,T]}$ is a continuous Q-martingale, the Itô formula implies that $\langle \ln w^i\rangle$ is the bounded variation part of the semi-martingale $(-2\ln w_t^i)_{t\in[0,T]}$. So,

$$\langle \ln w^i\rangle_t - \langle \ln w^i\rangle_s = -2E^Q[\ln w_t^i - \ln w_s^i|\mathcal{F}_s],$$

and we have

$$\sigma^i\big(\sigma^i - 2\lambda^i\big)E^Q\int_0^T g_t(w)\mathrm{d}t \\ \le E^Q\int_0^T g_t(w)\mathrm{d}\langle \ln w^i\rangle_t \\ \le \sigma^i\big(\sigma^i + 2\lambda^i\big)E^Q\int_0^T g_t(w)\mathrm{d}t,$$

for any bounded continuous and non-anticipating function $g : [0,T]\times\mathbb{D}(\mathbb{R}^{1+d}) \to \mathbb{R}_+$. This proves (3.1.3). The property (3.1.4) is a direct consequence of Lemma 3.2.4. In view of [22], Corollary VI.6.30, we have the convergence of the (sub)sequence

$$\mathscr{L}\left(\big(M^n, \langle M^n\rangle\big)\big|\, Q^n\right) \to \mathscr{L}\left((w, \langle w\rangle)|\, Q\right).$$

Since L^n is the d-dimensional martingale corresponding to the stochastic logarithm of M^n, we observe that

$$\big\langle L^{ni}, L^{nj}\big\rangle = \big(M_-^{ni}M_-^{nj}\big)^{-1}\cdot\big\langle M^{ni}, M^{nj}\big\rangle.$$

From (3.2.12) follows the tightness of the sequence

$$\mathscr{L}(((M^{ni}M^{nj})^{-1}, \langle M^{ni}, M^{nj}\rangle)|Q^n).$$

We deduce the convergence of the stochastic integrals, see [22], Th. VI.6.22 with VI.6.6, and we get

$$\mathscr{L}\big(\big\langle L^n\big\rangle\big|Q^n\big) \to \mathscr{L}\big(\big\langle \ln w\big\rangle\big|Q\big).$$

Corollary 3.2.4 implies that

$$\big\langle L^{ni}, L^{nj}\big\rangle = O\big(n^{-1/2}\big),$$

and (3.1.4) follows. □

Construction of dual martingales

In this paragraph, we shall show that each probability measure of $\mathcal{Q}$ can be approximated in the sense of Lemma 3.2.6. More precisely, we approximate the probability measures of a subset $\tilde{\mathcal{Q}}$ of $\mathcal{Q}$. These probability measures are characterized by a convenient representation. Nevertheless, Lemma 3.2.7 below shows that this restriction is not fundamental.

We define $\tilde{\mathcal{Q}}$ the following subset of $\mathcal{Q}$.

Let g be a non-anticipating bounded function on $[0,T] \times \mathbb{D}(\mathbb{R}^d)$ with values in the set of the real $d \times d$ matrices such that gg' is diagonal and

$$0 \vee \sigma^i(\sigma^i - 2\lambda^i) + \delta \leq (gg')^{ii} \leq \sigma^i(\sigma^i + 2\lambda^i) - \delta, \tag{3.2.22}$$

for some $\delta > 0$, and for some $\kappa > 0$,

$$|g_t^{ij}(\alpha) - g_s^{ij}(\beta)| \leq \kappa(|t-s| + ||\alpha - \beta||_T), \tag{3.2.23}$$

for $i, j \leq d,\ t, s \in [0,T],\ v, w \in \mathbb{D}(\mathbb{R}^d)$. Let B be a d-dimensional Brownian motion under a probability R. Define the d-dimensional R-martingale N with the components

$$N^i = \mathcal{E}(g^i(B) \cdot B). \tag{3.2.24}$$

For such processes N, we define $\tilde{\mathcal{Q}}$ the set of laws $\mathscr{L}((1,N)|R)$ on $\{1\} \times \mathbb{C}(\mathbb{R}^d)$.

The following lemma states that the laws of $\mathcal{Q}$ can be approximated by the laws of $\tilde{\mathcal{Q}}$ in a certain sense.

Lemma 3.2.7 *Let $Q \in \mathcal{Q}$ and consider the standard representation as in Remark 3.1.1. Namely, let B a Brownian motion under a probability R and g a process such that*

$$\mathscr{L}\left(\left(1, \mathcal{E}(g^1 \cdot B), \cdots, \mathcal{E}(g^d \cdot B)\right) \middle| R\right) = Q.$$

There exists a sequence of matrix-valued non-anticipating continuous bounded functions $(g^m)_{m \in \mathbb{N}}$ with $g^m g^{m\prime}$ diagonal and satisfying the conditions (3.2.22) (3.2.23) with the property

$$E^R \max_{1 \leq i \leq d} \left|\left|\mathcal{E}(g^i \cdot B) - \mathcal{E}(g^{mi}(B) \cdot B)\right|\right|_T \to 0, \qquad 1 \leq i \leq d.$$

Proof. According to Section 3.4, we can construct such a sequence of functions $(g^m)_{m\in\mathbb{N}}$ with $\sum_i (g^{mij})^2 \geq c^{mi} > 0$ and

$$E^R \int \left| g_t^{ij} - g_t^{mij}(B) \right|^2 \mathrm{d}t \to 0.$$

The result follows using the Burkholder–Davis–Gundy inequality. $\square$

Now, a constructive way of approximating the laws of $\tilde{\mathcal{Q}}$ is given in the following lemma. We shall use the notations

$$\operatorname{diag}\sigma = \begin{pmatrix} \sigma^1 & & 0 \\ & \ddots & \\ 0 & & \sigma^d \end{pmatrix}, \qquad \xi_k = \begin{pmatrix} \xi_k^1 \\ \vdots \\ \xi_k^d \end{pmatrix}.$$

Lemma 3.2.8 *Let $Q \in \tilde{\mathcal{Q}}$. There exists a sequence $Z^n \in \mathcal{M}^n$ such that,*

$$\mathscr{L}\left(M^n \middle| Q^n\right) \to Q,$$

where $M^n = Z^n/Z^{n0}$ and $Q^n = Z_T^{n0} P$.

Proof. Following the above definition of $\tilde{\mathcal{Q}}$, consider the Brownian motion B under the probability R and the process g such that $\mathscr{L}((1,N)|R) = Q$, where N is given by (3.2.24).

Let

$$K_{t_k}^n = \frac{1}{2}\left(g_{t_k}^n g_{t_k}^{n\,\prime} \operatorname{diag}\sigma^{-1} - \operatorname{diag}\sigma \right),$$

$$g_{t_k}^n = g_{t_{k-1}}\left(\sum_{l=0}^{k-1} \Delta B_{t_l}^n I_{[t_l,\infty[} \right),$$

$$\Delta B_{t_k}^n = (g_{t_k}^n)^{-1} \Delta L_{t_k}^n, \qquad \Delta L_{t_k}^{ni} = (M_{t_{k-1}}^{ni})^{-1} \Delta M_{t_k}^{ni}.$$

Note that $g_{t_k}^n$ is invertible since $g_{t_k}^n g_{t_k}^{n\,\prime}$ is symmetric positive definite.

For every $n \geq 1$, we define the $d+1$-dimensional process M^n whose zero component is equal identically to unit while others are constant on each interval $[t_k, t_{k+1}[$ with

$$M_{t_k}^{ni} = S_{t_k}^{ni}\left(1 + \sqrt{T/n} K_{t_k}^{nii} \xi_k^i \right), \qquad 1 \leq i \leq d.$$

Using the bounds (3.2.22) we easily deduce that for every $i \geq 1$ and sufficiently small $\delta > 0$ we have the inequalities

$$1 - \lambda^{ni} + \sqrt{T/n}\frac{\delta}{2\sigma} \leq M^{ni}_{t_k} \leq 1 + \lambda^{ni} - \sqrt{T/n}\frac{\delta}{2\sigma} \leq 1 + \lambda^{ni}.$$

For sufficiently large n

$$\frac{1}{1+\lambda^{ni}} = \frac{1}{1+\sqrt{T/n}\lambda^i} \leq 1 - \sqrt{T/n}\lambda^i + \sqrt{T/n}\frac{\delta}{2\sigma} = 1 - \lambda^{ni} + \sqrt{T/n}\frac{\delta}{2\sigma}$$

and we conclude from the resulting bounds that the process M^n takes values in $\widehat{K}^{\lambda^n *} \setminus \{0\}$ for large n and $M^{ni} = \mathcal{E}((g^n \cdot B^n)^i)$ for $i \geq 1$.

To compute the martingale measure of M^n, we need the expression of the stochastic logarithm of M^{ni} for $i \geq 1$,

$$\Delta L^{ni}_{t_k} = \frac{M^{ni}_{t_k}}{M^{ni}_{t_{k-1}}} - 1 = \frac{(1+\mu^{ni}+\sigma^{ni}\xi^i_k)\left(1+\sqrt{T/n}K^{nii}_{t_k}\xi^i_k\right)}{1+\sqrt{T/n}K^{nii}_{t_{k-1}}\xi^i_{k-1}} - 1.$$

After simple transformation we have:

$$\begin{aligned}\Delta L^{ni}_{t_k} = \frac{\sqrt{T/n}}{1+\sqrt{T/n}K^{nii}_{t_{k-1}}\xi^i_{k-1}} &\Big(\left(\sigma^i + K^{nii}_{t_k} + \mu^{ni}K^{nii}_{t_k}\right)\xi^i_k \\ &+ \sqrt{T/n}(\mu^i + \sigma^i K^{nii}_{t_k}) - K^{nii}_{t_{k-1}}\xi^i_{k-1}\Big).\end{aligned}$$

It is easily seen that M^n and, a fortiori, B^n are Q^n-martingales for the probability measure

$$Q^n = \mathcal{E}_T(q^n)P = \prod_{k=1}^{n}(1+\Delta q^n_{t_k})P,$$

$$\Delta q^n_{t_k} = -\sum_{i=1}^{d}\frac{\sqrt{T/n}(\mu^i + \sigma^i K^{nii}_{t_k}) - K^{nii}_{t_{k-1}}\xi^i_{k-1}}{\sigma^i + K^{nii}_{t_k} + \mu^{ni}K^{nii}_{t_k}}\xi^i_k.$$

The following formula defines the process Z^n which is a (strictly) consistent price system in the nth model:

$$Z^n_t = E\left[\mathcal{E}_T(q^n)|\,\mathcal{F}^n_{t_k}\right]M^n_{t_k}, \qquad t \in [t_k, t_{k+1}[.$$

It remains to check the convergence of the sequence $\mathscr{L}(M^n|Q^n)$ to Q. This will be deduced from the converge in law of the processes B^n to a Brownian motion. So we shall compute the quadratic characteristics of B^n and use a version of the central limit theorem in the Skorohod space.

Note that

$$\begin{aligned} E^n\left[\Delta B^n_{t_k}\Delta B^{n\,\prime}_{t_k}\middle|\,\mathcal{F}^m_{t_{k-1}}\right] &= E\left[(1+\Delta q^n_{t_k})\Delta B^n_{t_k}\Delta B^{n\,\prime}_{t_k}\middle|\,\mathcal{F}^n_{t_{k-1}}\right] \\ &= (g^n_{t_k})^{-1}E^n\left[\Delta L^n_{t_k}\Delta L^{n\,\prime}_{t_k}\middle|\,\mathcal{F}^n_{t_{k-1}}\right](g^n_{t_k})^{-1\prime}. \end{aligned}$$

By virtue of Lemma 3.2.4,

$$\sup_{k\le n}\left|E^n[\Delta L^{ni}_{t_k}\Delta L^{nj}_{t_k}|\mathcal{F}^n_{t_{k-1}}]\right| = O(n^{-3/2}), \quad i\neq j.$$

It remains to compute $E^n[\Delta L^{ni2}_{t_k}|\mathcal{F}^n_{t_{k-1}}]$, $i\le d$. We use the following estimations in order to simplify computations below. According to (3.2.23), we have:

$$||\Delta g^n||_T = O(n^{-1/2}), \qquad ||\Delta K^n||_T = O(n^{-1/2}).$$

Moreover, using the Taylor expansion formulae leads to the relations

$$\sup_{k\le n}\left|(1+\Delta q^n_{t_k}) - \left(1+[(\operatorname{diag}\sigma + K^n_{t_k})^{-1}K^n_{t_k}\xi_{k-1}]\xi_k\right)\right| = O(n^{-1/2}), \quad (3.2.25)$$

and

$$\sup_{k\le n}\left|\Delta L^n_{t_k} - \sqrt{T/n}\left[(\operatorname{diag}\sigma + K^n_{t_k})\xi_k - K^n_{t_k}\xi_{k-1}\right]\right| = O(n^{-1}). \qquad (3.2.26)$$

Note that we used the matrix form of the processes for the sake of a simplified presentation. We make the following estimate:

$$\begin{aligned} \sup_{k\le n}\Bigg| & E[(1+\Delta q^n_{t_k})(\Delta L^{ni}_{t_k})^2|\mathcal{F}^n_{t_{k-1}}] \\ & - (T/n)E\left[\left(1+(\sigma^i+K^{nii}_{t_k})^{-1}K^{nii}_{t_k}\xi^i_{k-1}\xi^i_k\right)\right. \\ & \left.\left((\sigma^i+K^{nii}_{t_k})\xi^i_k - K^{nii}_{t_k}\xi^i_{k-1}\right)^2\middle|\,\mathcal{F}^n_{t_{k-1}}\right]\Bigg| = O(n^{-3/2}). \end{aligned}$$

After a direct computation, we get an explicit formula for the approximating term in the above expression:

$$\begin{aligned} E & \left[\left(1+(\sigma^i+K^{nii}_{t_k})^{-1}K^{nii}_{t_k}\xi^i_{k-1}\xi^i_k\right)\left((\sigma^i+K^{nii}_{t_k})\xi^i_k - K^{nii}_{t_k}\xi^i_{k-1}\right)^2\middle|\,\mathcal{F}^n_{t_{k-1}}\right] \\ = & (\sigma^i+K^{nii}_{t_k})^2 - K^{nii2}_{t_k}. \end{aligned}$$

We obtain the key relation

$$E^n[\Delta B^n_{t_k} \Delta B^n_{t_k}{}' | \mathcal{F}^n_{t_{k-1}}] = \frac{T}{n} I_d + R^n_k, \tag{3.2.27}$$

with the family of matrices $\{R^n_k,\ k \leq n,\ n \in \mathbb{N}\}$ satisfying

$$\sup_{i,j \leq d,\ k \leq n} |R^{nij}_k| = O(n^{-3/2}).$$

Finally, note that

$$g_{t_k}(B^n) = g^n_{t_{k+1}}.$$

According to the Central Limit Theorem, [22], Theorem VIII.3.33, we have $\mathscr{L}(B^n|Q^n) \to \mathscr{L}(B|R)$, by virtue of (3.2.27) and the estimation $||\Delta B^n||_T = O(n^{-1/2})$, which implies the conditional Lindeberg condition. We deduce that

$$\mathscr{L}(B^n, g(B^n)|Q^n) \to \mathscr{L}(B, g(B)|R).$$

Set

$$X^n = g_-(B^n) \cdot B^n, \qquad X = g(B) \cdot B.$$

By virtue of [22], Th. VI.6.22 with Cor. VI.6.30, it follows that the above stochastic integrals converge in law. We get the convergence

$$\mathscr{L}\left(X^n, [X^n], \sum \Phi(\Delta X^n) \Big| Q^n\right) \to \mathscr{L}(X, [X], 0|R), \tag{3.2.28}$$

where Φ is the $\mathbb{R}^d$-valued function defined by

$$\Phi^i(x) = \ln(1 + x^i) - x^i + \frac{(x^i)^2}{2}.$$

It remains to check that the convergence described in (3.2.28) implies the convergence of the stochastic exponential. Since each limit process is continuous, we can study the convergence of each coordinate separately. We then refer to the following Lemma 3.2.9. The result is proved. □

Lemma 3.2.9 *Let X^n, X be scalar adapted processes where X is continuous and such that*

$$\mathscr{L}\left(X^n, [X^n], \sum \Phi_2(\Delta X^n) \Big| Q^n\right) \to \mathscr{L}\left(X, [X], 0 | Q\right),$$

with $\Phi_2(x) = \ln(1+x) - x + \frac{x^2}{2}$*. Then we have the following convergence in law of stochastic exponentials:*

$$\mathscr{L}(\mathcal{E}(X^n)|Q^n) \to \mathscr{L}(\mathcal{E}(X)|Q).$$

Proof. The claim follows by observing that

$$\mathcal{E}(X) = G\left(X, [X], \sum \Phi_2(\Delta X)\right),$$

with

$$G(x, y, z) = \exp\left(x - \frac{y}{2} + z\right).$$

Since G is continuous on $(\mathbb{D}(\mathbb{R}^3), ||.||_T)$, we get the result. □

3.3 Proof of the main result

We shall prove Theorem 3.1.2 using the sequential version of the definition of the closed convergence.

Preliminary remarks

We start with some general remarks and tools which link the technical ideas from Section 3.2 with superhedging issues.

First, observe that for any $Z \in \mathcal{M}^n$ we have the two-side inequalities

$$\frac{1}{1+|\lambda^n|} \le Z_0^i \le 1 + |\lambda^n| \tag{3.3.29}$$

and

$$\frac{1}{1+|\lambda^n|} S^{ni} \le Z^i/Z^0 \le (1+|\lambda^n|) S^{ni}. \tag{3.3.30}$$

In the following lines, we link superhedging and the particular convergence described in Lemmata 3.2.6 and 3.2.8. Let $Z^n \in \mathcal{M}^n$ be such that

$$\mathscr{L}\left(M^n \middle| Q^n\right) \to Q,$$

for $M^n := Z^n/Z^{n0}$ and $Q^n := Z_T^{n0} P$.

Lemma 3.3.1 *The sequence S^n is Q^n-tight and*

$$E^n M_T^n G(S^n) \to E^Q w_T G(w),$$

for any bounded continuous function $G : \mathbb{D}(\mathbb{R}^{d+1}) \to \mathbb{R}^{d+1}$.

Proof. The inequalities (3.3.30) imply the following two bounds,

$$||\ln S^n||_T \leq \ln(1+\lambda^{ni}) + ||\ln M^n||_T \,,$$

and

$$||S^n - M^n||_T \leq ||M^n||_T \, O(n^{-1/2}). \tag{3.3.31}$$

Hence, the sequence S^n is Q^n-tight since the sequence M^n is. Let $G : \mathbb{D}(\mathbb{R}^{d+1}) \to \mathbb{R}^{d+1}$ be a bounded continuous function. Fix $\varepsilon > 0$. Then there exists a compact set of $\mathbb{D}(\mathbb{R}^{d+1})$, such that

$$Q^n\big(M^n \in K^\varepsilon, S^n \in K^\varepsilon\big) \geq 1 - \varepsilon.$$

Take a sequence of Lipschitz functions G_m convergent to G pointwise. This convergence is uniform on compacts. in particular, $|G - G_m| \leq \varepsilon$ on K^ε for sufficiently large m.

We have, using the Cauchy–Schwarz inequality:

$$\begin{aligned} E^n\big|M_T^n\big|\big|G(M^n) - G(S^n)\big| &\leq 2\varepsilon^{1/2} \max|G| \left(E^n\big|\big|M^n\big|\big|_T^2\right)^{1/2} \\ &+ 2\varepsilon E^n\big|\big|M^n\big|\big|_T + C_m^\varepsilon E^n\big|\big|M^n\big|\big|_T \, ||S^n - M^n||_T \,, \end{aligned}$$

where the Lipschitz constant C_m^ε does not depend on n. Taking the limit in n we get, in virtue of (3.3.31) and Lemma 3.2.2 that the limit of the left-hand side is smaller than ε multiplied by a constant. Since ε is arbitrary, the lemma is proven. □

It follows from Lemma 3.3.1 that for any $v \in \mathbb{R}^{1+d}$,

$$EZ_T^n(F(S^n) - v) \to E^Q\left(w_T F(w) - \mathbf{1}v\right) \tag{3.3.32}$$

since we have

$$EZ_T^n(F(S^n) - v) = E^n M_T^n(F(S^n) - v).$$

We end the preliminaries paragraph by observing that for each v, $\delta > 0$, $Z^n \in \mathcal{M}^n$, we have

$$EZ_T^n(F(S^n) - (v + \delta\mathbf{1})) \leq EZ_T^n(F(S^n) - v) - \frac{d}{1 + |\lambda^n|}\delta.$$

The financial meaning of this inequality is obvious: larger initial investment in all assets helps to hedge the European option.

Proof of Theorem 3.1.2

(i) Fix $v \in \Gamma$. We have to find a subsequence $v^n \in \Gamma^n$ such that $v^n \to v$. To this end, choose $Z^n \in \mathcal{M}^n$ such that

$$EZ_T^n(F(S^n) - v) + \frac{1}{n} \geq \sup_{Z \in \mathcal{M}^n} EZ_T(F(S^n) - v).$$

By virtue of Lemma 3.2.6, eventually applied to a subsequence of (Z^n), there exists $Q \in \mathcal{Q}$ such that

$$\limsup_n EZ_T^n(F(S^n) - v) = E^Q\left(w_T F(w) - \mathbf{1}v\right) \leq 0.$$

It follows that there is a positive sequence $\delta^n \to 0$ such that

$$EZ_T^n(F(S^n) - v) \leq \delta^n.$$

Increasing the initial capital v to v^n where

$$v^n = v + (1 + |\lambda^n|)\frac{1}{d}\left(\delta^n + \frac{1}{n}\right)\mathbf{1},$$

we get the desired sequence $v^n \in \Gamma^n$ such that $v^n \to v$.

(ii) Show that for a convergent (sub)sequence $(v^n)_{n\in\mathbb{N}}$, $v^n \in \Gamma^n$, its limit v belongs to Γ. Let $\varepsilon > 0$. By virtue of Lemma 3.2.7, we can choose $Q \in \tilde{\mathcal{Q}}$ such that

$$E^Q\left(w_T F(w) - \mathbf{1}v\right) \geq \sup_{Q \in \mathcal{Q}} E^Q\left(w_T F(w) - \mathbf{1}v\right) - \varepsilon.$$

According to Lemma 3.2.8, together with (3.3.32), there is a sequence $Z^n \in \mathcal{M}^n$ such that

$$\liminf_n EZ_T^n(F(S^n) - v) = E^Q\left(w_T F(w) - \mathbf{1}v\right).$$

We conclude that this quantity is nonpositive using the boundedness of Z_0^n, (3.3.29). Indeed,

$$\liminf_n EZ_T^n(F(S^n) - v) = \liminf_n EZ_T^n(F(S^n) - v^n) + \liminf_n Z_0^n(v^n - v) \leq 0.$$

Since ε is arbitrary, v belongs to Γ. This ends the proof. $\square$

Remark 3.3.2 *In [30], the value of interest in* Γ^n *is the minimal initial endowment in money (with a zero position in any stocks) needed to hedge the option, i.e.*

$$x^n = \sup_{Z \in \mathcal{M}^n} EZ_T F(S^n) = \min \left\{ v^0 : \ v \in \Gamma^n \cap \mathbb{R}_+ e^0 \right\}.$$

It is easily seen that this quantity converge to

$$x = \sup_{Q \in \mathcal{Q}} E^Q w_T F(w) = \min \left\{ v^0 : \ v \in \Gamma \cap \mathbb{R}_+ e^0 \right\}.$$

We refer to Theorem 4.2.2 below for more information.

3.4 Appendix

In this section, we give the sketch of the approximation of the integrand process g in Remark 3.1.1 by the integrand processes of interest in $\tilde{Q}$ involved in Lemma 3.2.7. The first Lemma gives argument for the approximation with "Lipschitz" function satisfying (3.2.23), the second explains how to restrict the bounds of $(gg')^{ii}$ as in (3.2.22).

Lemma 3.4.1 *Let* B *a Brownian motion under a probability* R *and* $\mathbf{F}$ *the filtration generated by the process* B*. Let* g *be a scalar bounded* $\mathbf{F}$*-adapted process. There exists a sequence of non-anticipating bounded functions* $(g^m)_{m \in \mathbb{N}}$ *on* $[0,T] \times \mathbb{D}(\mathbb{R}^d)$ *satisfying the conditions*

$$\inf |g| \leq |g^m| \leq \sup |g|, \tag{3.4.33}$$

$$|g_t^m(\alpha) - g_s^m(\beta)| \leq \kappa_m(|t-s| + ||\alpha - \beta||_T), \tag{3.4.34}$$

for some $\kappa_m > 0$, such that

$$E^R \int |g_t - g_t^m(B)|^2 \,\mathrm{d}t \to 0.$$

Proof. We introduce the notation for $\alpha \in \mathbb{D}(\mathbb{R}^d)$:

$$\alpha_0^t = \alpha \mathbb{I}_{[0,t]} + \alpha_t \mathbb{I}_{]t,T]}.$$

The approximations hold in the following steps. Fix $\varepsilon > 0$.

- There exist n_0 and Borelian functions g^n on $(\mathbb{D}, d)$ such that:

  $$E^R \int \left| g_t - \sum_0^{n_0} g^n(B_0^{t_n}) \mathbb{I}_{]t_n, t_{n+1}]} \right|^2 \mathrm{d}t \leq \varepsilon,$$

  see Th. 4.41 in [1]. Moreover, since g is bounded, we can suppose that g^n are uniformly bounded by a constant C_g and we have

  $$|g^n - g^m| \leq K|t_n - t_m|,$$

  where $K = 2(n_0/T)C_g$.

- According to Th. 4.33 in [1], each g^n is (everywhere) pointwise limit of continuous functions on $(\mathbb{D}, d)$. Invoking Cor. 3.15 in [1], such a function is (everywhere) pointwise limit of sequences of Lipschitz functions on $(\mathbb{D}, d)$ with the same bounds as g^n. It follows that there exists some Lipschitz functions $\tilde{g}^n$ such that,

  $$E^R \int \left| g_t - \sum \tilde{g}^n(B_0^{t_n}) \mathbb{I}_{]t_n, t_{n+1}]} \right|^2 \mathrm{d}t \leq 2\varepsilon.$$

  So, each $\tilde{g}^n$ is Lipschitz on $(\mathbb{D}, d)$.

- We set, for $\delta > 0$ small enough

  $$f_t(\alpha) = \tilde{g}^n(\alpha_0^{t_n}), \qquad t \in [t_n + \delta, t_{n+1}]$$

  and use a linear interpolation on $[t_n, t_n + \delta]$. Thus, we have f non-anticipating and

  $$|f_t(\alpha) - f_s(\alpha)| \leq K_\delta |t - s|,$$

  for some constant K_δ depending on δ.

Now, we can verify that f satisfies (3.4.34). Set κ the biggest of the above Lipschitz constants and $t \leq s$, we have

$$|f_t(\alpha) - f_s(\beta)| \leq \kappa|t-s| + |f_t(\alpha) - f_t(\beta)|.$$

Since $f(t,\cdot)$ is κ-lipschitzienne, we have

$$|f_t(\alpha) - f_t(\beta)| \leq \kappa d(\alpha_0^t, \beta_0^t) \leq \kappa\|\alpha - \beta\|_T.$$

Since ε is arbitrary, choosing a sequence $\varepsilon^m \to 0$, one can get the sequence g^m of interest iterating the procedure. Which prove the result.□

Remark 3.4.2 *Because the approximation of the matrix valued process g in Remark 3.1.1 by g^m is defined componentwise, $g^m g^{m\prime}$ is not necessary diagonal. Nevertheless, for m large enough, we can find a Lipschitz orthogonal matrix valued function close to the identity matrix such that $(Mg^m)(Mg^m)'$ is diagonal.*

Lemma 3.4.3 *Let g be real $d \times d$ matrix such that gg' is diagonal and*

$$0 < c^i \leq (gg')^{ii} \leq C^i.$$

There exists a sequence g^n of $d \times d$ matrices such that $g^n g^{n\prime}$ is diagonal and

$$c^i + \delta^n \leq (g^n g^{n\prime})^{ii} \leq C^i - \delta^n,$$

for some decreasing sequence $\delta_n > 0$, $\delta_n \to 0$, such that $|g - g^n| \to 0$.

Proof. Fix $n > 0$, suppose that

$$C^1 - \delta_n \leq (gg')^{11} \leq C^1.$$

There exists $\varepsilon_{\delta_n} > 0$ such that

$$(1 - \varepsilon_{\delta_n})(gg')^{11} \leq C^1 - \delta_n.$$

Set

$$g^n = diag \begin{pmatrix} \sqrt{1 - \varepsilon_{\delta_n}} \\ 1 \\ \vdots \\ 1 \end{pmatrix} g.$$

It is easily seen that we have $\varepsilon_{\delta_n} \to 0$. The argument extend to other coordinate and for the lower bounds.$\square$

Chapter 4

Ramification

It is easily seen that for the case $d = 1$, our model is essentially the same as that of [30] and, hence, contains some novelty even for the model with one risky asset, see Remark 3.3.2 above. Inspecting the proofs above one can observe that the arguments still work when

$$\lambda^n = O(n^{-1/2}), \qquad \mu^n = O(n^{-1}), \qquad \sigma^n = O(n^{-1/2}).$$

One can easily extend the reasoning to non symmetric transaction costs, see Chapter 4.2 below.

4.1 General Models

In the case $d \geq 2$, the considered cones K^n correspond to a model of stock market where all transactions pass through the money. Nevertheless, it provides some information also for more general models. Namely, let us consider as an example the family of models of currency markets given by transaction cost matrices $\Lambda^n = \Lambda\sqrt{T/n}$, where the solvency cones are

$$K(\Lambda^n) = \text{cone}\left\{\left(1 + \sqrt{T/n}\lambda^{ij}\right) e^i - e^j,\ e^i,\ 0 \leq i, j \leq d\right\}.$$

Note that we can embed our models into currency markets with transaction costs matrices

$$\Lambda(\lambda^n) = \begin{pmatrix} 0 & \lambda^{n1} & \cdots & \lambda^{nd} \\ \lambda^{n1} & 0 & \lambda^n_{ij} & \\ \vdots & \lambda^n_{ij} & \ddots & \\ \lambda^{nd} & & & 0 \end{pmatrix},$$

where the transaction costs penalize direct exchanges, that is

$$1 + \lambda^n_{ij} \geq (1 + \lambda^{ni})(1 + \lambda^{nj}).$$

This remark leads to the following asymptotic bounds:

Proposition 4.1.1 *With obvious notations, we have the following inclusions:*

$$\Gamma\big(\overline{\lambda}\big) \subseteq \mathrm{Li}\ \Gamma^n\big(\Lambda^n\big) \subseteq \mathrm{Ls}\ \Gamma^n\big(\Lambda^n\big) \subseteq \Gamma\left(\underline{\lambda}\right),$$

where

$$\begin{aligned} \underline{\lambda}^i &= \max\{\lambda^i : \ (\Lambda(\lambda) - \Lambda)^{ij} \leq 0, \quad (\Lambda(\lambda) - \Lambda)^{ji} \leq 0 \quad j \neq i\}, \\ \overline{\lambda}^i &= \min\{\lambda^i : \ (\Lambda(\lambda) - \Lambda)^{ij} \geq 0, \quad (\Lambda(\lambda) - \Lambda)^{ji} \geq 0 \quad j \neq i\}. \end{aligned}$$

4.2 Non Symmetric Transaction costs

In this section, we concisely sum up argument to explain how to deal with non symmetric transaction costs in Section 3 (or in the paper [16]). The presentation may sightly differ. Indeed we shall detail with care the links between the traditional argument used in [30] with the geometric approach of the more involved paper [16], or equivalently Section 3, restricted to two-asset models.

4.2.1 Model and main result

We consider 2-asset models of currency market with transaction costs following the ideas of the book [24]. The first non-risky asset will serve as the numéraire, the second is risky. An asset can be exchanged to the other paying the proportional transaction costs. That is to increase the value of the jth position in one unit (of numéraire), one need to diminish in $(1+\lambda^{ij})$ unit (of numéraire) the ith position. Namely, the models are given by transaction costs matrices. We fix as basic parameter the 2-square matrix Λ with zero diagonal and positive entries. We consider the transaction cost matrix for the n-th model

$$\Lambda^n = \Lambda\sqrt{T/n}.$$

Price processes

We define in this subsection continuous-time models whose price processes are piecewise constant on the intervals forming uniform partitions of $[0,T]$. Of course, these models are in one-to-one correspondence with discrete–time models. Fix the drift and volatility parameters $\mu \in \mathbb{R}$, $\sigma \in]0,\infty[$ and put, for $n \geq 1$,

$$\mu^n = \mu T/n, \qquad \sigma^n = \sigma\sqrt{T/n}.$$

On the probability space $(\Omega, \mathcal{F}, P)$, we consider, for each n, a family of i.i.d. random variables $\{\xi_k;\ k \leq n\}$, where ξ_k take values in $\{-1,1\}$ and $P(\xi_k = 1) = 1/2$. Put

$$t_k = t_k^n := kT/n.$$

The process S^{n2} models the price evolution of one unit of the risky security measured in units of the first non-risky asset serving as numéraire. We define the process $S_t^n = (S_t^{n1}, S_t^{n2})$ where $S_t^{n1} = 1$ and

$$S_0^{n2} = 1, \qquad S_t^{n2} = \prod_{m=1}^{k} \left(1 + \mu^n + \sigma^n \xi_m\right), \qquad t \in [t_k, t_{k+1}[,$$

for sufficiently large n (to insure that $S^{n2} > 0$). In this setting the stochastic basis is $(\Omega, \mathcal{F}, \mathbf{F}^n, P)$ where the filtration $\mathbf{F^n} = (\mathcal{F}_t^n)$ is $\mathcal{F}_t^n := \sigma\{S_r^n,\ r \leq t\}$.

Transaction costs

The solvency region is the cone defined by

$$K^{\Lambda^n} = \text{cone}\left\{\left(1+\lambda^{n12}\right)e^1 - e^2,\ \left(1+\lambda^{n21}\right)e^2 - e^1\right\},$$

that is K^{Λ^n} is the set of positions which can be converted, paying transaction costs, to get only non-negative amount on each asset. The (positive) dual cone is the set

$$K^{\Lambda^n *} = \left\{w \in \mathbb{R}^2 : \frac{1}{1+\lambda^{n21}} \leq \frac{w^2}{w^1} \leq 1+\lambda^{n12}\right\},$$

which is the set of vectors with a non-negative scalar product with any vector of K^{Λ^n}.

The piecewise constant process V solving the linear controlled stochastic equation

$$V_0 = v \in K^{\Lambda^n}, \quad dV_t^i = V_{t-}^i dS_t^{ni}/S_{t-}^{ni} + dB_t^i, \qquad i = 1, 2,$$

models the portfolio value process with strategy B, where the components of the control B are

$$B^i = \sum_{k=1}^{n} B_k^i \mathbb{I}_{]t_{k-1},t_k]},$$

B_k^i is $\mathcal{F}_{t_{k-1}}^n$-measurable and $\Delta B_{t_k} = B_{t_k} - B_{t_{k-1}} \in L^0(-K^{\Lambda^n}, \mathcal{F}_{t_{k-1}}^n)$. The set of such processes V with initial value v is denoted by $\mathcal{A}_v^n$ while the notation $\mathcal{A}_v^n(T)$ is reserved for the set of their terminal value V_T.

Using the random diagonal operator

$$\phi_t^n : (x^1, x^2) \mapsto (x^1, x^2/S_t^{n2})$$

define the random cone $\widehat{K}_t^{\Lambda^n} = \phi_t^n K^{\Lambda^n}$ with the dual $\widehat{K}_t^{\Lambda^n *} = (\phi_t^n)^{-1} K^{\Lambda^n *}$.

Hedging sets

Our aim is to price a European option. We shall consider a two-dimensional pay-off. The first asset is an amount of money in numéraire, whereas the second is a quantity of physical units. The pay-off is of the form $F(S^n)$ with the function $F : \mathbb{D}(\mathbb{R}^2) \to \mathbb{R}^2_+$ supposed to be bounded and continuous in the Skorohod topology on $\mathbb{D}(\mathbb{R}^2)$. Let Γ^n be the set of initial endowments from which one can start a self-financing portfolio process with the terminal value dominating the contingent claim $F(S^n)$, that is

$$\Gamma^n = \{v \in \mathbb{R}^2 : \ (\phi^n_T)^{-1} F(S^n) \in \mathcal{A}^n_v(T) \ \text{ a.s.}\}.$$

We denote by $\mathcal{M}^n$ the set of all $\mathbf{F}^n$-martingales Z such that $Z_t \in \widehat{K}^{\Lambda^n *}_t \setminus \{0\}$ a.s. and $Z^1_0 = 1$. According to [24], Chap. 3,

$$\Gamma^n = \left\{v \in \mathbb{R}^2 : \ vZ_0 \geq EZ_T F(S^n) \ \text{ for all } Z \in \mathcal{M}^n\right\}. \tag{4.2.1}$$

This identity is the so-called hedging theorem claiming that one can super replicate the contingent claim if and only if the value of the initial endowments is not less than the expectation of the value of the contingent claim whatever a consistent price system is used to the comparison. The theorem holds under the assumption of the existence of a strictly consistent price system, fulfilled for our models.

Limit sets and main results

In analogy with the use of consistent price systems for the hedging theorem, we shall define the following set of martingales. Let B be a Brownian motion. We define $\mathcal{M}$ as the set of processes $(1, M)$,

$$M = \mathcal{E}(g \cdot B),$$

where g is a predictable adapted process whose square admits the following bounds:

$$\sigma(\sigma - 2\lambda) \leq g^2 \leq \sigma(\sigma + 2\lambda),$$

with λ be the mean of the transaction costs coefficients,

$$\lambda = \frac{\lambda^{12} + \lambda^{21}}{2}.$$

We put

$$\Gamma = \left\{v \in \mathbb{R}^2 : \ vZ_0 \geq EZ_T F(Z) \ \text{ for all } Z \in \mathcal{M}\right\}.$$

The main results of this note are the following. In the formulation of Theorem 4.2.1 below, we could refer to convergence in the closed topology of the subsets of $\mathbb{R}^2$, see [20]. We provide a simple but equivalent characterization in terms of sequences.

Theorem 4.2.1 *We have the convergence results,*

(i) for any $v \in \Gamma$, there is a sequence $v^n \in \Gamma^n$, such that $v^n \to v$,

(ii) for any convergent subsequence of the sequence $v^n \in \Gamma^n$, the limit belongs to Γ.

We also give the following auxiliary result. In [30], the value of interest in Γ^n is following:

$$x^n = \min \left\{ v^1 : \ v \in \Gamma^n \cap \mathbb{R}_+ e^1 \right\}.$$

This is the minimal initial capital with a zero position in the risky asset which hedge the option.

Theorem 4.2.2 *The sequence $\{x_n\}$ converges to x where*

$$x = \min \left\{ v^1 : \ v \in \Gamma \cap \mathbb{R}_+ e^1 \right\}.$$

4.2.2 Weak convergence

We obtain our convergence result for Γ^n by using the representation (4.2.1) and the theory of weak convergence of measures. In order to make argument more transparent, it is useful to consider a family of rather simpler polyhedral conic models in the spirit of Part I (or paper [17]). Indeed, there exists a sequence of positive numbers $\kappa^n = O(n^{-1/2})$ such that $K^{\Lambda^n *} \subset K^{\kappa^n *}$, where

$$K^{\kappa *} := \mathbb{R}_+(\mathbf{1} + U_\kappa) \qquad U_\kappa := \{v \in \mathbb{R}^2 : \ |v| \le \kappa\}.$$

That is, $K^{\kappa *}$ is the closed convex cone in $\mathbb{R}^2$ generated by the max-norm ball of radius κ with center at $\mathbf{1}$.

Let a sequence $Z^n \in \mathcal{M}^n$. It is easily seen that Z^n takes values in the cone $(\phi^n)^{-1}K^{\kappa^n *}$. The strictly positive martingale Z^{n1} is the density process of the probability measure $Q^n = Z^{n1}_T P$ and the process $M^n := Z^{n2}/Z^{n1}$ is a strictly positive Q^n-martingale with respect to the filtration $\mathbf{F}^n$. Observe that

$$\frac{1-\kappa^n}{1+\kappa^n} S^{n2} \leq M^n \leq \frac{1+\kappa^n}{1-\kappa^n} S^{n2}. \tag{4.2.2}$$

We shall show that the sequence M^n is Q^n-tight.

It is worth to note that there is a one-to-one correspondence between $\mathcal{M}^n$ and the set of "preconsistent price systems" of Kusuoka [30], it is particularly clear with the proposition 2.14 therein.

Let us define the piecewise constant processes ("stochastic logarithms" of M^n)

$$L^n := (M^n_-)^{-1} \cdot M^n.$$

Note that L^n has jumps only at the points t_k,

$$\Delta L^n_{t_k} = (M^n_{t_k-})^{-1} \Delta M^n_{t_k} = (M^n_{t_{k-1}})^{-1}(M^n_{t_k} - M^n_{t_{k-1}}), \qquad k \geq 1.$$

Tightness

The following lemma collects the basic asymptotics needed to check the tightness of the laws $\mathscr{L}(M^n|Q^n)$ on the Skorohod space.

Lemma 4.2.3 *We have the following asymptotic relations:*

$$||\Delta \ln M^n||_T = O(n^{-1/2}), \tag{4.2.3}$$

$$||\Delta L^n||_T = O(n^{-1/2}), \tag{4.2.4}$$

$$||\Delta \ln M^n - \Delta L^n||_T = O(n^{-1}), \tag{4.2.5}$$

$$\sup_{k \leq n} \left| E^{Q^n}[\Delta \ln M^n_{t_k} | \mathcal{F}^n_{t_{k-1}}] \right| = O(n^{-1}). \tag{4.2.6}$$

Proof. We derive from (4.2.2) the bounds

$$
\begin{aligned}
-2\ln\frac{1+\kappa^n}{1-\kappa^n} + \ln\left(1+\mu^n-\sigma^n\right) & \\
&\leq \Delta \ln M^n \\
&\qquad \leq 2\ln\frac{1+\kappa^n}{1-\kappa^n} + \ln\left(1+\mu^n+\sigma^n\right),
\end{aligned}
$$

implying (4.2.3). In view of the relation

$$\Delta L^n_{t_k} = \exp(\Delta \ln M^n_{t_k}) - 1,$$

we get (4.2.4). Setting

$$\Phi_1(z) := \ln(1+z) - z = O(z^2), \qquad z \to 0,$$

the asymptotic

$$||\Phi_1(\Delta L^n)||_T = O(n^{-1})$$

is a consequence of (4.2.4). Note that

$$\Delta \ln M^n_{t_k} = \Delta L^n_{t_k} + \Phi_1\left(\Delta L^n_{t_k}\right),$$

and (4.2.5), (4.2.6) follows. □

Lemma 4.2.4 *Let $Z^n \in \mathcal{M}^n$, $M^n := Z^{n2}/Z^{n1}$, and $Q^n := Z^{n1}_T P$. Then:*

(i) the sequence M^n is Q^n-C-tight;

(ii) the sequence S^n is Q^n-tight and

$$\left|\left|S^{n2} - M^n\right|\right|_T \leq ||M^n||_T\, O(n^{-1/2}). \tag{4.2.7}$$

Proof. Following the lines of Section 3 (or [16]) or [30], Lemma 4.8, we get bounds for the processes M^n and their bracket's oscillations. That is, for any $m > 1$, we have

$$\sup_n E^{Q^n}||M^n||^{2m}_T < \infty \qquad \text{and} \qquad \sup_n E^{Q^n}||\ln M^n||^{2m}_T < \infty, \tag{4.2.8}$$

and the following estimate for the increments of quadratic characteristics:

$$E^{Q^n} \sup_{k \le n-l} \left| \langle M^n \rangle_{t_{k+l}} - \langle M^n \rangle_{t_k} \right|^2 \le C(l/n)^2, \qquad l \le n, \tag{4.2.9}$$

where the constant C does not depend on l, n. The tightness of the sequence $\mathscr{L}(M^n|Q^n)$ follows, see [22]. Furthermore, we can deduce from Lemma 4.2.3 that the jumps tend to zero, which shows that each limit point of the sequence of laws $\mathscr{L}(M^n|Q^n)$ is continuous by virtue of Proposition VI.3.26 in [22].

From (4.2.2), we easily deduce (4.2.7) and the following,

$$|| \ln S^{n2} ||_T \le \ln \frac{1+\kappa^n}{1-\kappa^n} + ||\ln M^n||_T \, .$$

Which proves the second assertion. □

Identification of the limit laws

In this paragraph, we show that each limit law of the sequence $\mathscr{L}(Z^n/Z^{n1}|Q^n)$ is the law of a process in $\mathcal{M}$. With the definition of the processes of $\mathcal{M}$, one can see that we need an estimation of the quadratic variation process of L^n. This is the aim of Lemma 4.2.5 below.

Lemma 4.2.5 *We have the following asymptotic relations:*

$$-2E^{Q^n}[\ln M^n_{t_{k+l}} - \ln M^n_{t_k} | \mathcal{F}^n_{t_k}] \le (l/n)T\sigma(\sigma + 2\lambda) + R_n, \qquad l \le n, \; k \le n-l,$$

$$-2E^{Q^n}[\ln M^n_{t_{k+l}} - \ln M^n_{t_k} | \mathcal{F}^n_{t_k}] \ge (l/n)T\sigma(\sigma - 2\lambda) - R_n, \qquad l \le n, \; k \le n-l,$$

where the positive sequence $R_n = O(n^{-1/2})$ does not depend on k and l.

Proof. The proof of the lemma stands on the following two estimations:

$$\sup_{k \le n} \left| E^{Q^n}[2(\Delta \ln M^n_{t_k}) + (\Delta \ln M^n_{t_k})^2 | \mathcal{F}^n_{t_{k-1}}] \right| \quad = \quad O(n^{-3/2}), \tag{4.2.10}$$

$$\begin{aligned} \sup_{k \le n} \Big| E^{Q^n}[(\Delta \ln M^n_{t_k} + Y^n_{t_{k-1}})^2 - (Y^n_{t_k})^2 | \mathcal{F}^n_{t_{k-1}}] & \\ -\sigma^n(\sigma^n + 2E^{Q^n}[Y^n_{t_k}\xi_k | \mathcal{F}^n_{t_{k-1}}]) \Big| \quad &= \quad O(n^{-3/2}), \end{aligned} \tag{4.2.11}$$

where

$$Y^n := \ln M^n - \ln S^{n2} - \frac{\lambda^{n12} - \lambda^{n21}}{2}.$$

We start proving (4.2.10). Define the function

$$\Phi_2(z) := \ln(1+z) - z + z^2/2 = O(z^3), \qquad z \to 0.$$

We get the following obvious identity:

$$\begin{aligned} 2\Delta \ln M^n_{t_k} - 2\Delta L^n_{t_k} + (\ln M^n_{t_k} - E^{Q^n}[\ln M^n_{t_k}|\mathcal{F}^n_{t_{k-1}}])^2 \\ = 2\Phi_2(\Delta L^n_{t_k}) + (\ln M^n_{t_k} - E^{Q^n}[\ln M^n_{t_k}|\mathcal{F}^n_{t_{k-1}}] - \Delta L^n_{t_k})^2 \\ + 2\Delta L^n_{t_k}(\ln M^n_{t_k} - E^{Q^n}[\ln M^n_{t_k}|\mathcal{F}^n_{t_{k-1}}] - \Delta L^n_{t_k}). \end{aligned}$$

Due to Lemma 4.2.3, we have the following asymptotics

$$\begin{aligned} \sup_{k\le n}\left|\ln M^n_{t_k} - E^{Q^n}[\ln M^n_{t_k}|\mathcal{F}^n_{t_{k-1}}] - \Delta L^n_{t_k}\right| &= O(n^{-1}), \\ ||\Phi_2(\Delta L^n)||_T &= O(n^{-3/2}). \end{aligned}$$

Using this, we get

$$\sup_{k\le n}\left|E^{Q^n}[2(\Delta \ln M^n_{t_k}) + (\ln M^n_{t_k} - E^{Q^n}[\ln M^n_{t_k}|\mathcal{F}^n_{t_{k-1}}])^2|\mathcal{F}^n_{t_{k-1}}]\right| = O(n^{-3/2}).$$

This relation in conjunction with (4.2.3) and (4.2.6), gives us the first asymptotic bound (4.2.10).

We recall the following bounds

$$-\lambda^{n21} \le -\ln(1+\lambda^{n21}) \le \ln M^n - \ln S^{n2} \le \ln(1+\lambda^{n12}) \le \lambda^{n12}.$$

Using this, we obtain that

$$||Y^n||_T \le \lambda^n, \tag{4.2.12}$$

where $\lambda^n = \sqrt{T/n}\,\lambda$. By the relation

$$Y^n_{t_{k-1}} + \Delta \ln M^n_{t_k} = Y^n_{t_k} + \ln\big(1 + \mu^n + \sigma^n \xi_k\big),$$

we get the second main relation (4.2.11).

Now, we use (4.2.10) and (4.2.11) to complete the proof. With the expression

$$
\begin{aligned}
&2\Delta \ln M^n_{t_k} + \Delta(Y^n_{t_k})^2 \\
&\qquad = [2\Delta \ln M^n_{t_k} + (\Delta \ln M^n_{t_k})^2] - [(\Delta \ln M^n_{t_k} + Y^n_{t_{k-1}})^2 - (Y^n_{t_k})^2] \\
&\qquad\qquad + 2Y^n_{t_{k-1}} \Delta \ln M^n_{t_k},
\end{aligned}
$$

we deduce from (4.2.6), (4.2.10), (4.2.11), and (4.2.12) the key relation

$$
\begin{aligned}
\sup_{k\le n} \Big| & E^{Q^n}[2\Delta \ln M^n_{t_k} + \Delta(Y^n_{t_k})^2 | \mathcal{F}^n_{t_{k-1}}] \\
& + \sigma^n(\sigma^n + 2E^{Q^n}[Y^n_{t_k}\xi_k | \mathcal{F}^n_{t_{k-1}}]) \Big| = O(n^{-3/2}).
\end{aligned}
$$

It remains to observe that

$$
|2\sigma^n Y^n_{t_k}\xi_k| \le 2\sigma^n\lambda^n, \qquad k \le n.
$$

Hence there exists a positive constant κ such that

$$
\begin{aligned}
&- l\sigma^n(\sigma^n + 2\lambda^n) - \kappa l n^{-3/2} \\
&\qquad \le 2E^{Q^n}[\ln M^n_{t_{k+l}} - \ln M^n_{t_k} | \mathcal{F}^n_{t_k}] + E^{Q^n}[(Y^n_{t_{k+l}})^2 | \mathcal{F}^n_{t_k}] - (Y^n_{t_k})^2 \\
&\qquad\qquad \le -l\sigma^n(\sigma^n - 2\lambda^n) + \kappa l n^{-3/2}.
\end{aligned}
$$

Using (4.2.12) and the inequality $ln^{-3/2} \le n^{-1/2}$, we get

$$
\begin{aligned}
&- l\sigma^n(\sigma^n + 2\lambda^n) - \kappa n^{-1/2} - (\lambda^n)^2 \\
&\qquad \le 2E^{Q^n}[\ln M^n_{t_{k+l}} - \ln M^n_{t_k} | \mathcal{F}^n_{t_k}] \\
&\qquad\qquad \le -l\sigma^n(\sigma^n - 2\lambda^n) + \kappa n^{-1/2} + (\lambda^n)^2.
\end{aligned}
$$

This completes the proof. □

Lemma 4.2.6 *Let $Z^n \in \mathcal{M}^n$ and let $Q^n := Z^{n1}_T P$. For each cluster point Q of the sequence $\mathscr{L}(Z^n/Z^{n1}|Q^n)$, there exists a process $Z \in \mathcal{M}$ with $Q = \mathscr{L}(Z)$.*

Proof. Setting $\tilde{Q}^n = \mathscr{L}((1, M^n)|Q^n)$, Lemma 4.2.4 asserts that each cluster point Q of the tight sequence $\tilde{Q}^n$ charges only $\{1\} \times \mathbb{C}(\mathbb{R})$. On this set, the canonical process $\{(1, w_t);\ t \in [0, T]\}$ is a martingale under Q with respect to its natural filtration because of (4.2.8), see [22]. We shall show that the quadratic characteristics of logarithm of its second component is absolute continuous (with respect to Lebesgue measure) Q-a.s., with the bounds

$$\sigma(\sigma - 2\lambda)\mathrm{d}t \le \mathrm{d}\langle \ln w \rangle_t \le \sigma(\sigma + 2\lambda)\mathrm{d}t. \tag{4.2.13}$$

Equivalently, since $\{w_t;\ t \in [0, T]\}$ is a Q-martingale, $\langle \ln w \rangle$ is the bounded variation part of the semi-martingale $\{-2 \ln w_t;\ t \in [0, T]\}$ and we show that

$$\begin{aligned}
&\sigma\big(\sigma - 2\lambda\big) E^Q \int_0^T g_t(w)\mathrm{d}t \\
&\qquad\qquad \le E^Q \int_0^T g_t(w)\mathrm{d}\langle \ln w \rangle_t \\
&\qquad\qquad\qquad\qquad \le \sigma\big(\sigma + 2\lambda\big) E^Q \int_0^T g_t(w)\mathrm{d}t,
\end{aligned}$$

for any function $g : [0, T] \times \mathbb{D}(\mathbb{R}) \to \mathbb{R}_+$ which is bounded, continuous in the product of the usual topology on $[0, T]$ and the Skorohod topology on $\mathbb{D}(\mathbb{R})$ and adapted, i.e. $g_t(w)$ is $\sigma\{w_s, s \le t\}$-measurable for any t. The claim follows from Lemma 4.2.5 and (4.2.8). We have :

$$\limsup_{n\to\infty} E^{\tilde{Q}^n} g_s(w)\big(- 2(\ln w_t - \ln w_s) - \sigma\big(\sigma + 2\lambda\big)(t - s)\big) \le 0,$$

and

$$\liminf_{n\to\infty} E^{\tilde{Q}^n} g_s(w)\big(- 2(\ln w_t - \ln w_s) - \sigma\big(\sigma - 2\lambda\big)(t - s)\big) \ge 0.$$

Which lead to

$$-2E^Q g_s(w)(\ln w_t - \ln w_s) \le E^Q g_s(w)\sigma\big(\sigma + 2\lambda\big)(t - s),$$

and

$$-2E^Q g_s(w)(\ln w_t - \ln w_s) \ge E^Q g_s(w)\sigma\big(\sigma - 2\lambda\big)(t - s).$$

Hence Q on $\mathbb{C}(\mathbb{R}^2)$ is such that the (continuous) martingale part of $\{\ln w_t;\ t \in [0, T]\}$ has a quadratic characteristic process $\langle \ln w \rangle$ satisfying

(4.2.13). From [28], Theorem 3.4.2, Q admits the following standard representation. There exist B, a standard Brownian motion under a probability ν, and an adapted process g such that

$$\sigma(\sigma - 2\lambda) \leq g^2 \leq \sigma(\sigma + 2\lambda), \qquad 1 \leq i \leq d,$$

and

$$\mathscr{L}\left(1, \mathcal{E}(g \cdot B) | \, \nu\right) = Q.$$

$\square$

Construction of discrete martingales

The aim of the following section is to show that processes of $\mathcal{M}$ can be approximated by consistent price systems in $\mathcal{M}^n$. The following lemma gives a constructive way of approximating the martingales of a subset of $\mathcal{M}$.

Lemma 4.2.7 *Let B be a Brownian motion. Let g be an adapted continuous bounded function : $[0,T] \times \mathbb{D}(\mathbb{R}) \to \mathbb{R}^+ \setminus \{0\}$ such that, for some $\delta > 0$,*

$$\delta \vee \sigma(\sigma - 2\lambda) + \delta \leq g^2 \leq \sigma(\sigma + 2\lambda) - \delta, \tag{4.2.14}$$

$$|g_t(w) - g_s(v)| \leq \kappa(|t - s| + ||w - v||_T), \tag{4.2.15}$$

for $t, s \in [0,T],\ v, w \in \mathbb{C}(\mathbb{R})$. Define the martingale

$$M = \mathcal{E}(g(B) \cdot B).$$

Then there exists a sequence $Z^n \in \mathcal{M}^n$ such that

$$\mathscr{L}(Z^n/Z^{1n}|Q^n) \to \mathscr{L}((1, M)|Q),$$

with $Q^n = Z_T^{1n} P$.

Proof. We consider the piecewise constant process

$$M_{t_k}^n = \frac{1 + 1/2\lambda^{n12}}{1 + 1/2\lambda^{n21}} \left(1 + K_{t_k}^n \sqrt{T/n}\xi_k\right) S_{t_k}^{n2}, \qquad 0 \leq k \leq n,$$

with K^n the predictable process defined by

$$\begin{aligned} K^n_{t_k} &= \frac{1}{2\sigma}(g^n_{t_k})^2 - \frac{\sigma}{2}, \\ g^n_{t_k} &= g_{t_{k-1}}\big((B^n_{t_l})_{l=0}^{k-1}\big), \end{aligned} \tag{4.2.16}$$

where the process B^n is piecewise constant with the jumps

$$\begin{aligned} \Delta B^n_{t_k} &= (g^n_{t_k})^{-1}\Delta L^n_{t_k}, \\ \Delta L^n_{t_k} &= (M^n_{t_{k-1}})^{-1}\Delta M^n_{t_k}. \end{aligned} \tag{4.2.17}$$

The proof consists in two steps. The first one is to construct from M^n a sequence of consistent price systems in $\mathcal{M}^n$. The second step is to check the convergence.

According to (4.2.14),

$$-\lambda + \varepsilon \le K^n \le \lambda - \varepsilon,$$

for some $\varepsilon > 0$. Using the Taylor expansion formulae, we get the bounds

$$1 - \lambda^{n21} + \varepsilon R^1_n \le \frac{1 + 1/2\lambda^{n12}}{1 + 1/2\lambda^{n21}}\Big(1 + K^n_{t_k}\sqrt{T/n}\xi_k\Big) \le 1 + \lambda^{n12} - \varepsilon R^1_n,$$

where $R^1_n = O(n^{-1/2})$ and $R^1_n > 0$ for large n. It is easily seen that

$$\frac{1}{1+\lambda^{n21}}S^{n2} \le M^n \le (1+\lambda^{n12})S^{n2}$$

for sufficiently large n. These inequalities show that $(1, M^n)$ takes values in $\widehat{K}^{\Lambda^n *}\backslash\{0\}$ for sufficiently large n. Our aim now is to determine the martingale measure of M^n. We compute the stochastic logarithm of M^n,

$$\begin{aligned} \Delta L^n_{t_k} &= \frac{M^n_{t_k}}{M^n_{t_{k-1}}} - 1 \\ &= \frac{(1+\mu^n+\sigma^n\xi_k)(1+K^n_{t_k}\sqrt{T/n}\xi_k)}{1+K^n_{t_{k-1}}\sqrt{T/n}\xi_{k-1}} - 1 \\ &= \sqrt{\frac{T}{n}}\,\frac{(\sigma + K^n_{t_k} + \mu_n K^n_{t_k})\xi_k + \mu\sqrt{T/n} + \sigma_n K^n_{t_k} - K^n_{t_{k-1}}\xi_{k-1}}{1+\sqrt{T/n}K^n_{t_{k-1}}\xi_{k-1}}. \end{aligned}$$

Observe that M^n is a Q^n-martingale where Q^n is given by

$$Q^n = \mathcal{E}(q^n)_T P, \qquad \triangle q^n_{t_k} = -\frac{\mu\sqrt{T/n} + \sigma_n K^n_{t_k} - K^n_{t_{k-1}}\xi_{k-1}}{(\sigma + K^n_{t_k} + \mu_n K^n_{t_k})}\xi_k,$$

recalling that for a piecewise constant process q,

$$\mathcal{E}(q)_t = \prod_{s\le t}(1+\Delta q_s).$$

Setting

$$Z^n_t = E\left[\mathcal{E}(q^n)_T \middle| \mathcal{F}^n_{t_k}\right](1, M^n_{t_k}), \qquad t_k \le t < t_{k+1},$$

we get a sequence of martingales taking values in $\widehat{K}^{\Lambda^n *}$, that is a sequence of consistent price systems.

In view of (4.2.17), we have the expression

$$M^n = \mathcal{E}(g^n \cdot B^n).$$

We shall use a version of the Central Limit Theorem to show the convergence of $\mathscr{L}(B^n|Q^n)$ to the law of a Brownian motion. We need to compute the increments of the quadratic variation process of B^n, that is $E^{Q^n}[(\Delta B^n_{t_k})^2|\mathcal{F}^n_{t_{k-1}}]$. First, according to (4.2.15) and (4.2.17), observe that

$$||\Delta B^n||_T = O(n^{-1/2}), \qquad ||\Delta K^n||_T = O(n^{-1/2}).$$

It follows that

$$\sup_{k\le n}\left|\Delta L^n_{t_k} - \sqrt{T/n}\left[(\sigma + K^n_{t_k})\xi_k - K^n_{t_k}\xi_{k-1}\right]\right| = O\left(n^{-1}\right), \tag{4.2.18}$$

and

$$\sup_{k\le n}\left|\Delta q^n_{t_k} - \frac{K^n_{t_k}\xi_{k-1}\xi_k}{\sigma + K^n_{t_k}}\right| = O\left(n^{-1/2}\right). \tag{4.2.19}$$

Having in mind the expression

$$E^{Q^n}\left[(\Delta B^n_{t_k})^2 \middle| \mathcal{F}^n_{t_{k-1}}\right] = (g^n_k)^{-2} E[(1+\Delta q^n_{t_k})(\Delta L^n_{t_k})^2|\mathcal{F}^n_{t_{k-1}}],$$

we deduce from (4.2.18) and (4.2.19),

$$E^{Q^n}\left[(\Delta B^n_{t_k})^2 \middle| \mathcal{F}^n_{t_{k-1}}\right] = \frac{T}{n}(g^n_{t_k})^{-2}\left((\sigma + K^n_{t_k})^2 - (K^n_{t_k})^2\right) + R^n_{t_k},$$

where $||R^n||_T = O(n^{-3/2})$. Finally, with the definition of K^n, (4.2.16), it is easily seen that

$$E^{Q^n}\left[(\Delta B^n_{t_k})^2 \big| \mathcal{F}^n_{t_{k-1}}\right] = \frac{T}{n} + R^n_{t_k}.$$

Note also that the sequence B^n satisfies the conditional Lindeberg hypothesis, Property VIII.3.31 in [22]. By the use of the Central Limit Theorem, [22], VIII.3.33, we get the existence of a Brownian motion B such that

$$\mathscr{L}\big(B^n, g(B^n)\big|Q^n\big) \to \mathscr{L}(B, g(B)).$$

The announced convergence can be checked through the convergence of the stochastic exponential, and then the convergence of $\mathscr{L}(M^n|Q^n)$ to the law of the process M holds. □

Note that approximating processes of Lemma 4.2.7 allows us to approximate processes of $\mathcal{M}$. Indeed, let $Z \in \mathcal{M}$, $Z^2 = \mathcal{E}(g \cdot B)$. It is easily seen that we can construct a sequence of functions $(g^m)_{m\in\mathbb{N}}$ satisfying the assumptions of Lemma 4.2.7 with

$$E\int |g_t - g^m_t(B)|^2 \,\mathrm{d}t \to 0.$$

Using the Burkholder–Davis–Gundy inequality, we get that

$$E\left|\left|\mathcal{E}(g\cdot B) - \mathcal{E}(g^m_-(B)\cdot B)\right|\right|_T \to 0.$$

4.2.3 Proof of the main results

Preliminary remarks

We first give some general remarks and tools which link the technical ideas from Section 4.2.2 with super hedging issues.

Remind the assertion (4.2.2), that is for any $Z \in \mathcal{M}^n$,

$$\frac{1-\kappa^n}{1+\kappa^n} \le Z_0^2 \le \frac{1+\kappa^n}{1-\kappa^n}, \tag{4.2.20}$$

and, more generally,

$$\frac{1-\kappa^n}{1+\kappa^n}S^{n2} \le Z^2/Z^1 \le \frac{1+\kappa^n}{1-\kappa^n}S^{n2}.$$

Now we show that the particular convergence described in Lemmata 4.2.4 and 4.2.7 is consistent with the hedging theorem. Let $Z^n \in \mathcal{M}^n$ be such that for $M^n := Z^{n2}/Z^{n1}$ and $Q^n := Z^{n1}_T P$ we have

$$\mathscr{L}\left((1, M^n)|\, Q^n\right) \to \mathscr{L}(Z),$$

for some $Z \in \mathcal{M}$. It follows from Lemma 4.2.4.2 that for any $v \in \mathbb{R}^2$,

$$EZ^n_T(F(S^n) - v) \to EZ_T(F(Z) - v), \tag{4.2.21}$$

since

$$EZ^n_T(F(S^n) - v) = E^{Q^n}(1, M^n_T)(F(S^n) - v).$$

We end this paragraph observing the fact that increasing the initial capital both on the first and the second asset helps to hedge the European option. Indeed, for each v, $\delta > 0$, $Z^n \in \mathcal{M}^n$, we have

$$EZ^n_T(F(S^n) - (v + \delta \mathbf{1})) \leq EZ^n_T(F(S^n) - v) - 2\frac{1 - \kappa^n}{1 + \kappa^n}\delta. \tag{4.2.22}$$

Moreover, this bound is uniform on the choice of the consistent price system.

Proof of Theorem 4.2.2

The proof of this theorem is similar to the one given in [30]. Note that

$$x_n = \sup_{Z \in \mathcal{M}^n} EZ_T F(S^n),$$

and

$$x = \sup_{Z \in \mathcal{M}} EZ_T F(Z).$$

We proceed by establishing the following two inequalities:

$$\limsup_n x^n \leq x, \qquad \liminf_n x^n \geq x.$$

For the first one, we fix the sequence $Z^n \in \mathcal{M}^n$ such that

$$EZ^n_T F(S^n) \geq x^n - 1/n.$$

According to Lemmata 4.2.4 and 4.2.6, there exist a subsequence Z^{n_k} and a process $Z \in \mathcal{M}$ such that

$$\limsup_n EZ_T^n F(S^n) = \lim_k EZ_T^{n_k} F(S^{n_k}) = EZ_T F(Z) \leq x.$$

Conversely, we fix $\varepsilon > 0$ and choose $Z \in \mathcal{M}$ such that

$$EZ_T F(Z) \geq x - \varepsilon.$$

By virtue of Lemma 4.2.7, there exists a sequence $Z^n \in \mathcal{M}^n$ such that

$$\liminf_n EZ_T^n F(S^n) = EZ_T F(Z).$$

Since ε is arbitrary, we get $\liminf x^n \geq x$, and Theorem 4.2.2 is proved.

Proof of Theorem 4.2.1, Assertion 1

The proof of Theorem 4.2.1 follows the same reasoning based on choosing the best candidate between the consistent price systems. However, the fact that we consider convergence of sets makes the demonstration more involved. Here we prove the first assertion.

Fix $v \in \Gamma$, we shall construct a sequence $v^n \in \Gamma^n$ such that $v^n \to v$. Choose a sequence $Z^n \in \mathcal{M}^n$ such that

$$EZ_T^n(F(S^n) - v) + \frac{1}{n} \geq \sup_{Z \in \mathcal{M}^n} EZ_T(F(S^n) - v).$$

As a consequence of Lemmata 4.2.4 and 4.2.6, there exists $Z \in \mathcal{M}$ such that

$$\limsup_n EZ_T^n(F(S^n) - v) = EZ_T(F(Z) - v) \leq 0.$$

It follows that there is a positive sequence $\delta^n \to 0$ such that

$$EZ_T^n(F(S^n) - v) \leq \delta^n.$$

Define v^n by increasing the initial capital v to

$$v^n = v + \frac{1}{2}\frac{1+\kappa^n}{1-\kappa^n}\left(\delta^n + \frac{1}{n}\right)\mathbf{1}.$$

Having in mind (4.2.22), it is easily seen that for any $Z \in \mathcal{M}^n$, we have:

$$EZ_T(F(S^n)-v^n) \leq EZ_T(F(S^n)-v)-\left(\delta^n + \frac{1}{n}\right) \leq EZ_T^n(F(S^n)-v)-\delta^n \leq 0.$$

So we constructed the desired sequence $v^n \in \Gamma^n$ such that $v^n \to v$.

Proof of Theorem 4.2.1, Assertion 2

It remains to show that for a convergent (sub)sequence $v^n \in \Gamma^n$, the limit v belongs to Γ. Fix $\varepsilon > 0$ and choose $Z \in \mathcal{M}$ such that

$$EZ_T\left(F(Z) - v\right) \geq \sup_{Z \in \mathcal{M}} EZ_T\left(F(Z) - v\right) - \varepsilon.$$

By virtue of Lemma 4.2.7 and (4.2.21), there is a sequence $Z^n \in \mathcal{M}^n$ such that

$$\liminf_n EZ_T^n(F(S^n) - v) = EZ_T(F(Z) - v).$$

Note that

$$\liminf_n EZ_T^n(F(S^n) - v) = \liminf_n EZ_T^n(F(S^n) - v^n) + \liminf_n Z_0^n(v^n - v),$$

and

$$\liminf_n Z_0^n(v^n - v) = 0,$$

since Z_0^n is bounded, (4.2.20). We can conclude that

$$EZ_T(F(Z) - v) \leq 0$$

and since ε is arbitrary, v belongs to Γ. This ends the proof. $\square$

Part III

Approximative Hedging

Option pricing gathers finance industry needs and Quantitative Finance. The complexity of models increases to match the real world, as for example in the papers [5, 11, 21]. Simulation methods have to be developed with two aims, a good accuracy and a low computational cost. In the books [4, 29], many finite difference schemes are considered. They are simulated by Monte-Carlo Methods which compute the mean by generating a big number of asset price realizations. These methods suffer from the difficulty of generating the Brownian sample paths, since the discretization of the process implies a loss of accuracy.

Other methods study the density of the option price at the exercise date. This is the subject of the famous paper by Black–Scholes [2]. In the papers [13, 14], the asset price evolutions are approximated by Picard iterations. A scheme using an expansion with the Wiener–Ito Chaos formula is introduced. The density of the first three terms are then approximated. The accuracy of this method is illustrated by numerical simulations but not theoretically studied.

We use the so-called Picard iterations in a rigorous framework. We introduce a discretized scheme which can be simulated by Monte-Carlo methods. This studying part provides a very basic scheme to be compared with the Euler scheme. We first focus on the second term in the Picard iterations. In this case, the scheme is mainly relevant to ("generalized") European options. Even if a systematic error, in the spirit of the one in [13, 14], has to be accepted, we obtain a good convergence speed, namely n^{-1}. For the higher Picard iterations, though we loose the systematic error, the convergence speed is worse than $n^{-1/2}$. Further research has to study faster schemes.

This part is organized as follows. In the following section, we present the mainstream of the option hedging and we rigorously introduce the approximation of the asset price by Picard iterations and the discretization scheme. In a second section, we discuss the case of the second iteration, introducing assumption on the pay-off function. In a third section, we discuss about general case. A subsidiary section gathers some integrability properties and various tools.

Cette partie fait l'objet d'un article en préparation en coécriture avec Emmanuel Lépinette.

Chapter 5

Approximation by Picard Iterations

5.1 The Model

Let $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}, P)$ be a continuous–time stochastic basis satisfying the usual assumptions (in particular complete) supporting a standard Brownian motion W, i.e. $\mathcal{F}_t := \sigma(W_s : s \leq t) \vee \mathcal{N}$ where $\mathcal{N}$ is the family of all sets of P-measure zero.

According to Th. 2.2 in [12], p. 104, we have the following.

Proposition 5.1.1 *Suppose that $\sigma : [0,T]\times\mathbb{R}_+ \to \mathbb{R}$ and $r : [0,T]\times\mathbb{R}_+ \to \mathbb{R}$ are two Lipschitz functions. Then, the s.d.e.*

$$dS_t = S_t\sigma(t, S_t)dW_t + S_t r(t, S_t)dt, \quad S_0 = x,$$

has a unique strong solution.

In the sequel, we make the following assumption on the Lipschitz functions σ and r.

Assumption 5.1.2 *Assume that the function r is bounded and*

$$0 < \sigma^2(t, y) \leq L(1 + \ln(\ln(y))1_{y>1}), \qquad \forall t \in [0,T],\ \forall y \in \mathbb{R}_+.$$

Assumption 5.1.2 essentially stands for the existence of the moments of $\sup_{t\in[0,T]} S_t$, see Section 8.1.

5.1.1 The Option

The above process S models the price evolution of the underlying asset. We aim at approximating the valuation of the option with the pay-off $G(S)$ where $G : \mathbb{C}(\mathbb{R}_+) \to \mathbb{R}_+$ is supposed to satisfy :

$$|G(\alpha) - G(\beta)| \le C \sup_{t\in[0,T]} |\alpha_t - \beta_t|, \qquad \alpha, \beta \in \mathbb{C}(\mathbb{R}_+). \tag{5.1.1}$$

We assume, without restriction, that P is the risk-neutral measure so that the valuation of the contingent claim is $v = EG(S)$. This means that r has to be considered as the interest rate of the bank account. Hence, the following assumption on r is natural.

Assumption 5.1.3 *We suppose that $r(s, \cdot) =: r_s$ is deterministic and bounded.*

We set

$$F(t) = x \exp\left(\int_0^t r_u du\right).$$

For an integrand H, we write (when it does make sense)

$$S_t(H) = F(t) \exp\left(H \cdot W_t - \frac{1}{2}\int_0^t H_u^2 du\right).$$

For $n \in \mathbb{N}$, we set $\tau_n = \{t_0 = 0, t_1 = T/n, t_2 = 2T/n, \cdots t_n = T\}$ the uniform partition of the time interval $[0, T]$. For a process X, we denote by X^n the piecewise constant process

$$\begin{aligned} X_t^n &= X_{t_i}, \qquad t_i \le t < t_{i+1}, \\ X_T^n &= X_T. \end{aligned}$$

5.2 The General Approximation

The aim of the current section is to use an approximation $\tilde{S}$ of S in the valuation of the option. It is important to note that the theoretical convergence speed of the option approximation relies on the following norm of the difference between S and $\tilde{S}$. Since (5.1.1) holds, we have

$$|EG(S) - EG(\tilde{S})| \leq E|G(S) - G(\tilde{S})| \leq C \left\| \sup_{t \in [0,T]} \left| S_t - \tilde{S}_t \right| \right\|_2 .$$

That is, we always study the mean squared error when introducing a new scheme. The approximation of interest is obtained in three steps. The first one consists in bounding the volatility. The second one uses the so-called Picard iterations, a recursive scheme where we approximate the solution of the s.d.e. satisfied by S by the solution of s.d.e.'s with iterated (bounded) diffusion. The third step is the approximation of the diffusion by a discretization method, see Section 5.2.3 and Chapters 6, 7 below.

5.2.1 Bounded Diffusion

We first bound the diffusion process σ with the parameter κ. Let $\kappa \in \mathbb{R}_+$, with $\kappa \geq r_s$, $\forall s \in [0,T]$, and $\kappa > x = S_0$. Consider Y^κ the unique solution to the s.d.e.

$$\begin{aligned} Y_0^\kappa &= 0, \\ dY_t^\kappa &= \sigma(t, xe^{Y_t^\kappa} \wedge \kappa)dW_t + r_t dt - \frac{1}{2}\sigma^2(t, xe^{Y_t^\kappa} \wedge \kappa)dt. \end{aligned}$$

Note that $S^\kappa := xe^{Y^\kappa}$ satisfies

$$\begin{aligned} S_0^\kappa &= x \\ S_t^\kappa &:= x \exp\left[\int_0^t \sigma(u, S_u^\kappa \wedge \kappa)dW_u + \int_0^t r_u - \frac{1}{2}\sigma^2(u, S_u^\kappa \wedge \kappa)du \right]. \end{aligned}$$

Throughout the paper, we also denote Y such that $S = xe^Y$. The following lemmata state the convergence of S^κ to S.

Lemma 5.2.1 *S^κ converges pointwise on $[0,T]$ to S.*

Proof. Consider the stopping times

$$\tau^\kappa := \inf\{t:\ |S_t| \geq \kappa\} \wedge T.$$

Then, the stopped processes S^{τ^κ} and $(S^\kappa)^{\tau^\kappa}$ satisfy the same s.d.e.. It follows that $S_t = S_t^\kappa$ on $t \in [0, \tau^\kappa]$. As $\tau^\kappa \to \infty$, we conclude. □

Lemma 5.2.2 *Suppose that Assumption 5.1.2 holds, there are some constants C_p such that for all $p \geq 1$*

$$E \sup_{t\in[0,T]} (S_t)^{2p} + \sup_\kappa E \sup_{t\in[0,T]} (S_t^\kappa)^{2p} \leq C_p,$$

and therefore, for all $l \geq 1$ there are constants C_l such that

$$E \sup_{t\in[0,T]} (S_t^\kappa - S_t)^2 \leq \frac{C_l}{\kappa^l}.$$

Proof. The proof of the first property is postponed in Section 8.1. For the second one, observe that

$$\begin{aligned}
E \sup_{t\in[0,T]} \big(S_t^\kappa - S_t\big)^2 &= E \sup_{t\in[0,T]} \big(S_t^\kappa - S_t\big)^2 1_{\tau^\kappa < t} \\
&\leq E \sup_{t\in[0,T]} \big(S_t^\kappa - S_t\big)^2 1_{\tau^\kappa < T} \\
&\leq E \sup_{t\in[0,T]} \big(S_t^\kappa - S_t\big)^2 1_{\sup_{t\in[0,T]} S_t \geq \kappa} \\
&\leq \sqrt{E \sup_{t\in[0,T]} \big(S_t^\kappa - S_t\big)^4} \sqrt{\frac{E\big(\sup_{t\in[0,T]} S_t\big)^{2l}}{\kappa^{2l}}}.
\end{aligned}$$

Which yields the result. □

5.2.2 Picard Iterations

The idea of the following scheme is to construct successive solutions $S^{\kappa,m}$ of s.d.e.'s with iterated diffusion such that $S^{\kappa,m}$ converge to S^κ. To do so, we introduce

$$\begin{aligned} S^{\kappa,0} &= x, \\ S_t^{\kappa,m+1} &:= x \exp\left[\int_0^t \sigma\big(u, S_u^{\kappa,m} \wedge \kappa\big) dW_u + \int_0^t r_u - \frac{1}{2}\sigma^2\big(u, S_u^{\kappa,m} \wedge \kappa\big) du\right]. \end{aligned}$$

We set

$$Y_t^{\kappa,m} := \log S_t^{\kappa,m} - \log x.$$

As a matter of fact, this process satisfies the following s.d.e.

$$dY_t^{\kappa,m+1} = \sigma\big(t, xe^{Y_t^{\kappa,m}} \wedge \kappa\big) dW_t + r_t dt - \frac{1}{2}\sigma^2\big(t, xe^{Y_t^{\kappa,m}} \wedge \kappa\big) dt.$$

To this end, we use the short notation

$$\sigma_s^m := \sigma\big(s, xe^{Y_s^{\kappa,m-1}} \wedge \kappa\big) = \sigma\big(s, S_s^{\kappa,m-1} \wedge \kappa\big), \qquad m > 0.$$

The following lemmata state the convergence results of $S^{\kappa,m}$ to S^κ in L^2. We first focus on the fourth moments of the error between $Y^{\kappa,m}$ and Y^κ.

Lemma 5.2.3 *The sequence $Y^{\kappa,m}$ converges in L^2 to Y^κ such that*

$$\left\|\sup_{u \le t} \left|Y_u^\kappa - Y_u^{\kappa,m}\right|\right\|_2 \le C(1+|x|) \sum_{j=m}^{\infty} (\kappa C)^j \frac{\sqrt{T}^j}{\sqrt{j!}}, \tag{5.2.2}$$

$$\left\|\sup_{u \le t} \left|Y_u^\kappa - Y_u^{\kappa,m}\right|\right\|_4 \le C(1+|x|) \sum_{j=m}^{\infty} (\kappa C)^j \frac{(T^{1/4})^j}{(j!)^{1/4}}. \tag{5.2.3}$$

Proof. The following useful inequality is easily stated for $\alpha \ge 2$,

$$\left|x^{\alpha/2} - y^{\alpha/2}\right| \le \frac{\alpha}{2} \left(\max(|x|;|y|)\right)^{\alpha/2-1} |x-y|.$$

We deduce that

$$\left(\int_0^t \left(\sigma_s^{m+1}\right)^2 - \left(\sigma_s^m\right)^2 ds\right)^2 \le C \int_0^t \left(\sigma_s^{m+1} - \sigma_s^m\right)^2 ds,$$

using the Jensen inequality. Together with the Burkholder–Davis–Gundy inequalities, we obtain that

$$E \sup_{u \le t} \left|Y_u^{\kappa,m+1} - Y_u^{\kappa,m}\right|^2 \le CE \int_0^t \sup_{u \le s} \left|\sigma_u^{m+1} - \sigma_u^m\right|^2 ds.$$

Observe that $|\sigma(t, e^X \wedge \kappa) - \sigma(t, e^Y \wedge \kappa)| \leq \kappa C|X - Y|$, hence

$$\left|\sigma_u^{m+1} - \sigma_u^m\right| \leq \kappa C\left|Y_u^{\kappa,m} - Y_u^{\kappa,m-1}\right|, \qquad \forall u \in [0,T].$$

We get

$$E\sup_{u\leq t}\left|Y_u^{\kappa,m+1} - Y_u^{\kappa,m}\right|^2 \leq \kappa^2 C\int_0^t E\sup_{u\leq s}\left|Y_u^{\kappa,m} - Y_u^{\kappa,m-1}\right|^2 ds.$$

With Lemma 8.2.1, we deduce that

$$\begin{aligned} E\sup_{u\leq t}\left|Y_u^{\kappa,m+1} - Y_u^{\kappa,m}\right|^2 &\leq (\kappa^2 C)^m E\sup_{u\leq t}\left|Y_u^{\kappa,1} - Y_u^{\kappa,0}\right|^2 \frac{t^m}{m!} \\ &\leq C(\kappa^2 C)^m \frac{t^m}{m!}(1+x^2), \end{aligned} \tag{5.2.4}$$

where C does not depend on m. We deduce (5.2.2). Similarly we get (5.2.3) and the claim follows. □

It is worth to note that the mean squared error of (5.2.2) and (5.2.3) can be turned into the rest of exponential expansion series.

Corollary 5.2.4 *We have the following bounds*

$$E\sup_{u\leq t}\left|Y_u^{\kappa} - Y_u^{\kappa,m}\right|^2 \leq C(1+x^2)\sum_{j=m}^{\infty}\frac{(2\kappa^2 CT)^j}{j!},$$

$$E\sup_{u\leq t}\left|Y_u^{\kappa} - Y_u^{\kappa,m}\right|^4 \leq C(1+x^4)\sum_{j=m}^{\infty}\frac{(8\kappa^4 CT)^j}{j!}.$$

Proof. By virtue of Lemma 8.2.3, we get the inequality

$$E\sup_{u\leq t}\left|Y_u^{\kappa} - Y_u^{\kappa,m}\right|^2 \leq \sum_{j=m}^{\infty} 2^j E\sup_{u\leq t}\left|Y_u^{\kappa,j+1} - Y_u^{\kappa,j}\right|^2,$$

and we get the first inequality from (5.2.4) and similarly the second one. □

We use the above bounds to evaluate the mean squared error between S^κ and $S^{\kappa,m}$.

Lemma 5.2.5 *For every κ, there exists a constant C_κ such that*

$$E\sup_{t\in[0,T]}\left(S_t^\kappa - S_t^{\kappa,m}\right)^2 \leq C_\kappa\sqrt{E\sup_{t\in[0,T]}(Y_t^\kappa - Y_t^{\kappa,m-1})^4}.$$

Proof. By virtue of the Burkholder–Davis–Gundy inequalities, we have

$$E \sup_{u\in[0,t]} \left(S_u^\kappa - S_u^{\kappa,m}\right)^2 \leq CE\int_0^t \left(S_s^\kappa \sigma_s - S_s^{\kappa,m}\sigma_s^m\right)^2 ds + 2E\int_0^t \left(S_s^\kappa r_s - S_s^{\kappa,m} r_s\right)^2 ds,$$

where, by an abuse of notation, $\sigma_s := \sigma(s, S_s^\kappa \wedge \kappa)$, recalling that $\sigma_s^m := \sigma(s, S_s^{\kappa,m-1} \wedge \kappa)$. The first term is bounded from above as follows

$$E\int_0^T \left(S_s^\kappa \sigma_s - S_s^{\kappa,m}\sigma_s^m\right)^2 ds \leq 2E\int_0^T \left(S_s^\kappa(\sigma_s - \sigma_s^m)\right)^2 ds + 2E\int_0^T \left(\sigma_s^m(S_s^\kappa - S_s^{\kappa,m})\right)^2 ds.$$

Observe the inequalities $|\sigma(t,y)| + |r_t| \leq m_\kappa$ if $|y| \leq \kappa$ and

$$|\sigma(t, xe^X \wedge \kappa) - \sigma(t, xe^Y \wedge \kappa)| \leq \kappa C|X - Y|, \tag{5.2.5}$$

for $t \in [0,T]$. We recall that, according to Lemma 5.2.2

$$\sup_\kappa E \sup_{t\in[0,T]} (S_t^\kappa)^{2p} \leq C_p.$$

Therefore,

$$E \sup_{u\in[0,t]} \left(S_u^\kappa - S_u^{\kappa,m}\right)^2 \leq 6\kappa C\sqrt{E\sup_{u\leq T}\left|Y_u^\kappa - Y_u^{\kappa,m-1}\right|^4} + 6m_\kappa \int_0^t E\sup_{u\leq s}\left(S_u^\kappa - S_u^{\kappa,m}\right)^2 ds.$$

It remains to use Gronwall's Lemma to deduce that

$$E \sup_{t\in[0,T]} (S_t^\kappa - S_t^{\kappa,m})^2 \leq 6\kappa C\exp(6m_\kappa T)\sqrt{E\sup_{t\leq T}|Y_t^\kappa - Y_t^{\kappa,m-1}|^4}.$$

And the result follows. □

It is worth to mention, with the current notations, that we can write $S^{\kappa,m} = S(\sigma^m)$.

5.2.3 Approximation of the Diffusion

We shall now approximate σ by a recursive method. Here is the starting point of, we hope, various fruitful approximation methods. As an example, we consider a very simple discretization, considering the sample path of the Brownian motion only at a few dates. Though very basic, this method is known for being coarse. We shall discuss about its accuracy in the following two chapters. Fix $\kappa \in \mathbb{R}_+$. Set $\tau_n = \{t_0 = 0, t_1 = T/n, t_2 = 2T/n, \cdots t_n = T\}$ the uniform partition of order n of the time interval $[0, T]$. We define the following scheme. Suppose that $T/n \leq 1$. To alleviate notation, we write for the process σ

$$||\sigma||_t := \left(\int_0^t \sigma_r^2 dr \right)^{1/2}.$$

Consider the piecewise constant processes recursively defined by

$$\begin{aligned} \tilde{\sigma}_t^1 &= \sigma(t_i, x \wedge \kappa), \qquad t_i \leq t < t_{i+1}, \\ \tilde{\sigma}_t^m &= \sigma\left(t_i, F(t_i) \exp\left(\tilde{\sigma}^{m-1} \cdot W_{t_i} - \frac{1}{2} \|\tilde{\sigma}^{m-1}\|_{t_i}^2 \right) \wedge \kappa \right), \qquad t_i \leq t < t_{i+1}. \end{aligned}$$

Remark 5.2.6 *It is worth to mention that even if the processes $\tilde{\sigma}^m$ are piecewise constant, the processes $S(\tilde{\sigma}^m)$ are not. Indeed*

$$\begin{aligned} S_t(\tilde{\sigma}^m) &= F(t) \exp\left(\tilde{\sigma}^m \cdot W_t - \frac{1}{2} \int_0^t (\tilde{\sigma}_u^m)^2 du \right) \\ &= F(t) \exp\left(\tilde{\sigma}^m \cdot W_{t_i} + \tilde{\sigma}_{t_i}^m (W_t - W_{t_i}) - \frac{1}{2} \|\tilde{\sigma}^m\|_{t_i}^2 - \frac{1}{2} (\tilde{\sigma}_{t_i}^m)^2 (t - t_i) \right) \\ &= F(t) S_{t_i} \exp\left(\tilde{\sigma}_{t_i}^m (W_t - W_{t_i}) - \frac{1}{2} (\tilde{\sigma}_{t_i}^m)^2 (t - t_i) \right), \end{aligned}$$

for $t_i \leq t < t_{i+1}$. Nevertheless, it is possible to get simulations for the process $S^n(\tilde{\sigma}^m)$.

Chapter 6

The Case $m = 2$

In the case $m = 2$, even with the coarse approximation of the Brownian motion process, the accuracy is still good. Theorem 6.3.1 below reads as follows. Under assumptions on the pay-off function, accepting a certain non reducible error, the rate of convergence of the approximation is n^{-1}. In analogy with [13, 14], numerical simulations would illustrate the accuracy of the method. Nevertheless, we think that the systematic error is not much worse than the one in [13, 14].

6.1 Approximation

We shall approximate σ^2, the second iterated volatility defined in Paragraph 5.2.2, by the "recursive" discretization method of Paragraph 5.2.3. That is, we stop the iterations in Section 5.2.3 in the special case $m = 2$. Fix $\kappa \in \mathbb{R}_+$. We define the following scheme. Set $n \in \mathbb{N}$ and τ_n the sequence $\{t_i := iT/n\}$. We suppose that $T/n \leq 1$. Consider the piecewise constant processes defined by

$$
\begin{aligned}
\tilde{\sigma}^1_t &= \sigma(t_i, x \wedge \kappa), \qquad t_i \leq t < t_{i+1}, \\
\tilde{\sigma}^2_t &= \sigma\left(t_i, F(t_i) \exp\left(\tilde{\sigma}^1 \cdot W_{t_i} - \frac{1}{2}\|\tilde{\sigma}^1\|^2_{t_i}\right) \wedge \kappa\right), \qquad t_i \leq t < t_{i+1}.
\end{aligned}
$$

6.2 Consistency with Discretization

We introduce here a property on the pay-off function. Namely, we consider pay-off functions that suit well when the asset price is approximated by a piecewise constant process active only on a uniform partition of the time interval. We formulate the condition in a technical sense since we target, with further research, a larger class of pay-off functions, for example pay-off depending on the asset price at a few dates, etc. Further investigation on the pay-off functions are needed.

We need a condition which allows us to consider a "discretized" version of the underlying asset price, when this one is given by the Picard iterations. Set $n \in \mathbb{N}$ and τ_n the sequence $\{t_i := iT/n\}$. We say that the pay-off G is consistent with discretization if

$$|EG(S(\sigma^2)) - EG(S^n(\sigma^2))| \leq \frac{C_\kappa}{n}.$$

At least, the European call pay-off satisfies the consistency with discretization property. Indeed, the pay-off of the European call option with strike K is of the form $G(S)$ where $G(\alpha) = e^{-\int_0^T r_t dt}(\alpha_T - K)^+$. Since G depends only on the terminal value of α, we clearly have

$$G(S(\sigma^2)) = G(S^n(\sigma^2)).$$

6.3 Accuracy

The accuracy of our approximation is given in the following Theorem. One can see that with the above "discretization" property, the rate of convergence is higher than the one we could expect with the current approximation of the Brownian motion.

Theorem 6.3.1 *Assume that Assumptions 5.1.2 and 5.1.3 hold and suppose that G is consistent with discretization. Fix $\kappa \in \mathbb{R}_+$ and $n \in \mathbb{N}$. Then, for $l \geq 1$, there are some constants C_l, C_κ and $\epsilon_\kappa > 0$ such that*

$$|EG(S) - EG(S^n(\tilde{\sigma}^2))| \leq \frac{C_l}{\kappa^l} + \frac{C_\kappa}{n} + \epsilon_\kappa,$$

where $\varepsilon_\kappa := \varepsilon_{\kappa,2}$ *is the second term of a decreasing sequence* $\varepsilon_{\kappa,m} \to 0$ *as* $m \to \infty$*, see term (6.4.2) below.*

6.4 Proof of Theorem 6.3.1

To prove Theorem 6.3.1 we first remark that we have

$$\begin{aligned} |EG(S) - EG(S^n(\tilde{\sigma}^2))| \;\leq\; & |EG(S) - EG(S^\kappa)| & (6.4.1)\\ & +|EG(S^\kappa) - EG(S^{\kappa,2})| & (6.4.2)\\ & +|EG(S(\sigma^2)) - EG(S^n(\sigma^2))| & (6.4.3)\\ & +|EG(S^n(\sigma^2)) - EG(S^n(\tilde{\sigma}^2))|. & (6.4.4)\end{aligned}$$

Note that since G is consistent with discretization, the term (6.4.3) is smaller than CT/n. We recall that G is Lipschitz continuous by (5.1.1). That is we can evaluate the above terms summing the square root of the mean squared error studied before. Indeed (6.4.1) is bounded with the above Lemma 5.2.2, ϵ_κ stands for the quantity (6.4.2) and by the following Lemma 6.4.1 we bound (6.4.4). It is enough to study the convergence of $S^n(\tilde{\sigma}^2)$ to $S^n(\sigma^2)$.

Lemma 6.4.1 *We have the following inequality*

$$E \sup_{t\in[0,T]} \left|S_t^n(\tilde{\sigma}^2) - S_t^n(\sigma^2)\right|^2 \leq \frac{(\kappa CT)^2}{n^2} + \frac{C_l}{\kappa^l}.$$

Proof. Since the family of random variables $\{\tilde{\sigma}^1 \cdot W_t, \sigma^1 \cdot W_t, t \in [0,T]\}$ is uniformly integrable, there exists C such that, setting

$$\Gamma_C = \left\{\sup_{t\leq T} \sigma^1 \cdot W_t \geq C\right\} \bigcup \left\{\sup_{t\leq T} \tilde{\sigma}^1 \cdot W_t \geq C\right\}$$

we have $P(\Gamma_C) \leq C_l/\kappa^l$. It follows that

$$|S(\tilde{\sigma}^2)_{t_i} - S(\sigma^2)_{t_i}| I_{\Gamma_C^c} \leq C_{\kappa,l} \left| \left(\tilde{\sigma}^1 - \sigma^1\right) \cdot W_{t_i} - \frac{1}{2}\int_0^{t_i} \left(\tilde{\sigma}_s^1\right)^2 - \left(\sigma_s^1\right)^2 ds\right|.$$

With usual argument, we conclude that

$$\begin{aligned} E\max_i |S(\tilde{\sigma}^2)_{t_i} - S(\sigma^2)_{t_i}|^2 \quad &\leq \quad \sqrt{P(\Gamma_C)}\sqrt{E\sup_{t\in[0,T]}\left|S(\tilde{\sigma}^2)_t + S(\sigma^2)_t\right|^4} \\ &\quad +C_{\kappa,l}\int_0^T E\sup_{t\in[0,T]}\left|\tilde{\sigma}_t^1 - \sigma_t^1\right|^2 ds. \end{aligned}$$

It is easily seen that

$$\sup_{t\in[0,T]}\left|\tilde{\sigma}_t^1 - \sigma_t^1\right|^2 \leq \frac{(\kappa CT)^2}{n^2}. \tag{6.4.5}$$

Which ends the proof.□

Theorem 6.3.1 is proved.

Chapter 7

The Case $m > 2$

The good accuracy of the dicretization procedure fails to generalize when we consider higher iterations than the one described in the previous chapter. The main problem comes from the fact that (6.4.5) in the proof of Theorem 6.3.1 does not hold any more for $m > 2$. Then we suffer the lack of precision of the current approximation of the Brownian motion.

7.1 The Result

In view of the following proofs, it is hopeless to focus only on certain dates of the approximation of the asset price in order to improve the accuracy. So we do not need anymore the additional "discretization" property introduced in Paragraph 6.2. The accuracy of the approximation is stated in the following.

Theorem 7.1.1 *Assume that Assumptions 5.1.2 and 5.1.3 hold. Fix $\kappa \in \mathbb{R}_+$, $m \in \mathbb{N}$ and $n \in \mathbb{N}$. Then, for $l, p \in \mathbb{N}$, there are some constants $C_l, C, C_\kappa, C_{\kappa,p}$ such that*

$$|EG(S) - EG(S^n(\tilde{\sigma}^m))| \leq \frac{C_l}{\kappa^l} + C_\kappa \sqrt{\sum_{i=m-1}^{\infty} \frac{(8\kappa^4 C)^i}{i!}} + C_{\kappa,p} \left(\frac{1}{\sqrt{n}}\right)^{(p-1)/p}.$$

7.2 Proof of Theorem 7.1.1

In the spirit of the proof of Theorem 6.3.1, we make a survey of the mean squared errors. We have the following estimation.

Lemma 7.2.1 *There exists a constant $C_{\kappa,p}$ which does not depend on n and m such that*

$$\left\| \sup_{t\in[0,T]} |\tilde{\sigma}_t^m - \sigma_t^m| \right\|_p^p \leq \left(\frac{C_{\kappa,p}}{n} \right)^{(p-1)/2}.$$

Proof. Observe that

$$\begin{aligned} \left| \sup_{t\in[0,T]} \left|\tilde{\sigma}_t^m - \sigma_t^m\right| \right|^p \quad &\leq \quad 2^{p-1} \left| \max_{i\leq n} \left|\tilde{\sigma}_{t_i}^m - \sigma_{t_i}^m\right| \right|^p \\ &\quad +2^{p-1} \left| \max_{i\leq n} \sup_{t\in[t_i,t_{i+1}[} \left|\sigma_{t_i}^m - \sigma_t^m\right| \right|^p. \end{aligned} \tag{7.2.1}$$

So, we first evaluate the quantity $\|\tilde{\sigma}_t^m - \sigma_t^m\|_p^p$ at dates $\{t_i\}$. Since σ is Lipschitz, (5.2.5), there exists a constant C such that

$$\left|\tilde{\sigma}_{t_i}^m - \sigma_{t_i}^m\right| \leq \kappa C \left| \left(\tilde{\sigma}^{m-1} - \sigma^{m-1}\right) \cdot W_{t_i} - \frac{1}{2} \int_0^{t_i} \left[\left(\tilde{\sigma}_t^{m-1}\right)^2 - \left(\sigma_t^{m-1}\right)^2 \right] dt \right|.$$

We deduce that

$$\begin{aligned} \left\| \max_{i\leq n} \left|\tilde{\sigma}_{t_i}^m - \sigma_{t_i}^m\right| \right\|_p \leq \quad & \kappa C \left\| \max_{i\leq n} \left| \left(\tilde{\sigma}^{m-1} - \sigma^{m-1}\right) \cdot W_{t_i} \right| \right\|_p \\ & +\kappa C \left(E \left| \int_0^{t_i} \left(\tilde{\sigma}_t^{m-1}\right)^2 - \left(\sigma_t^{m-1}\right)^2 dt \right|^p \right)^{1/p}. \end{aligned}$$

Recall for $\alpha \geq 2$ the inequality

$$\left|x^{\alpha/2} - y^{\alpha/2}\right| \leq \frac{\alpha}{2} \left(\max(|x|;|y|)\right)^{\alpha/2-1} |x-y|.$$

Together with the Jensen inequality, we state that

$$E \left| \int_0^{t_i} \left(\tilde{\sigma}^{m-1}\right)^2 - \left(\sigma^{m-1}\right)^2 dt \right|^p \leq \kappa C E \int_0^{t_i} \left|\tilde{\sigma}_t^{m-1} - \sigma_t^{m-1}\right|^p dt.$$

It follows, using Burkholder–Davis–Gundy inequality, that we have

$$E\max_{i\le n}\left|\tilde{\sigma}_{t_i}^m-\sigma_{t_i}^m\right|^p \le (\kappa C_p)^p\int_0^{t_i} E\left|\tilde{\sigma}_t^{m-1}-\sigma_t^{m-1}\right|^p dt,$$

for some constant C_p depending on p.

In a second step, we study the second term in inequality (7.2.1). Using equality (5.2.5), we deduce that

$$\begin{aligned}
\sup_{t\in[t_i,t_{i+1}[}\left|\sigma_{t_i}^m-\sigma_t^m\right| &\le C\frac{T}{n}+\kappa C\sup_{t\in[t_i,t_{i+1}[}|Y_{t_i}^{\kappa,m-1}-Y_t^{\kappa,m-1}|,\\
\max_i\sup_{t\in[t_i,t_{i+1}[}\left|\sigma_{t_i}^m-\sigma_t^m\right|^p &\le \frac{2^{p-1}(CT)^p}{n^p}\\
&\quad +2^{p-1}\kappa^p C\sum_i\sup_{t\in[t_i,t_{i+1}[}|Y_{t_i}^{\kappa,m-1}-Y_t^{\kappa,m-1}|^p.
\end{aligned}$$

Once again with Burkholder–Davis–Gundy inequality, it follows that

$$E\max_i\sup_{t\in[t_i,t_{i+1}[}\left|\sigma_{t_i}^m-\sigma_t^m\right|^p \le \frac{2^{p-1}T^p}{n^p}+2^{p-1}C_p\sum_i(T/n)^{p/2}.$$

Therefore, we can conclude that

$$\begin{aligned}
E\left|\sup_{u\in[0,t]}|\tilde{\sigma}_u^m-\sigma_u^m|\right|^p &\le 2^{p-1}(\kappa C_p)^p\int_0^t E\sup_{u\in[0,s]}|\tilde{\sigma}_u^{m-1}-\sigma_u^{m-1}|^p ds\\
&\quad +\kappa^p C_p(T/n)^{(p-1)/2}.
\end{aligned}$$

Using Lemma 8.2.2, we deduce that

$$E\left|\sup_{t\in[0,T]}|\tilde{\sigma}_t^m-\sigma_t^m|\right|^p \le 2^{p-1}(\kappa C_p)^p\left(\frac{T}{n}\right)^{(p-1)/2}.$$

This ends the proof. □

As an evident corollary of the above Lemma 7.2.1, we state the following.

Corollary 7.2.2 *We have the following inequalities,*

$$\begin{aligned}\left\|\sup_{t\in[0,T]}\left|\sigma^m\cdot W_t-\tilde{\sigma}^m\cdot W_t\right|\right\|_p &\leq \frac{C_{\kappa,p}}{n^{(p-1)/(2p)}},\\ \left\|\sup_{t\in[0,T]}\left|\left\|\sigma^m\right\|_t^2-\left\|\tilde{\sigma}^m\right\|_t^2\right|\right\|_p &\leq \frac{C_{\kappa,p}}{n^{(p-1)/(2p)}}.\end{aligned}$$

We then prove the accuracy of the approximation of $S^{\kappa,m}$ by $S(\tilde{\sigma}^m)$ in the next lemma.

Lemma 7.2.3 *For every κ, there is a constant C_κ, which does not depend on m, such that*

$$\left\|\sup_{t\in[0,T]}\left|S_t^{\kappa,m}-S(\tilde{\sigma}^m)_t\right|\right\|_2 \leq \frac{C_{\kappa,p}}{n^{(p-1)/(2p)}}.$$

Proof. We follow the lines of the proof of Proposition 5.2.5. We have to observe that $S(\tilde{\sigma}^m)$ solves the s.d.e.

$$dS(\tilde{\sigma}^m)_t=S(\tilde{\sigma}^m)_t\tilde{\sigma}_t^m dW_t+S(\tilde{\sigma}^m)_t r_t dt.$$

It follows, by the Burkholder–Davis–Gundy inequality, that

$$\begin{aligned}E\sup_{u\in[0,t]}\left|S(\tilde{\sigma}^m)_u-S_u^{\kappa,m}\right|^2\leq \quad & CE\int_0^t\left(S(\tilde{\sigma}^m)_s\tilde{\sigma}_s^m-S_s^{\kappa,m}\sigma_s^m\right)^2 ds\\ &+2E\int_0^t\left(\left(S(\tilde{\sigma}^m)_s-S_s^{\kappa,m}\right)r_s\right)^2 ds.\end{aligned}$$

Focus on the first term of the above inequality. We have

$$\begin{aligned}E\int_0^t\left(S(\tilde{\sigma}^m)_s\tilde{\sigma}_s^m-S_s^{\kappa,m}\sigma_s^m\right)^2 ds \;\leq\; & 2E\int_0^t\left(S_s^{\kappa,m}\left(\tilde{\sigma}_s^m-\sigma_s^m\right)\right)^2 ds\\ &+2E\int_0^t\left(\tilde{\sigma}_s^m\left(S(\tilde{\sigma}^m)_s-S_s^{\kappa,m}\right)\right)^2 ds.\end{aligned}$$

Recall the inequalities

$$\begin{aligned}&|\sigma(t,x)|+|r_t|\leq m_\kappa, \qquad \text{if}|x|\leq\kappa,\\ &|\sigma(t,xe^X\wedge\kappa)-\sigma(t,xe^Y\wedge\kappa)|\leq\kappa C|X-Y|.\end{aligned}$$

We deduce that

$$
\begin{aligned}
E \sup_{u\in[0,t]} \left|S(\tilde{\sigma}^m)_u - S_u^{\kappa,m}\right|^2 \;\; &\leq \;\; \kappa C \left\| \sup_{t\in[0,T]} \left|\sigma^{m-1}\cdot W_t - \tilde{\sigma}^{m-1}\cdot W_t\right| \right\|_p^2 \\
&\quad + \kappa C \left\| \sup_{t\in[0,T]} \left| \left\|\sigma^{m-1}\right\|_t^2 - \left\|\tilde{\sigma}^{m-1}\right\|_t^2 \right| \right\|_p^2 \\
&\quad + 6 m_\kappa \int_0^t E \sup_{u\leq s} \left(S(\tilde{\sigma}^m)_u - S_u^{\kappa,m}\right)^2 ds.
\end{aligned}
$$

We conclude, using Gronwall's Lemma and Lemma 7.2.2. □

It remains to sum the errors in analogy with the proof of Theorem 6.3.1. Namely we have

$$
\begin{aligned}
|EG(S) - EG(S^n(\tilde{\sigma}^m))| \;\; &\leq \;\; |EG(S) - EG(S^\kappa)| && (7.2.2)\\
&\quad + |EG(S^\kappa) - EG(S^{\kappa,m})| && (7.2.3)\\
&\quad + |EG(S(\sigma^m)) - EG(S(\tilde{\sigma}^m))| && (7.2.4)\\
&\quad + |EG(S(\tilde{\sigma}^m)) - EG(S^n(\tilde{\sigma}^m))|. && (7.2.5)
\end{aligned}
$$

We recall that G is Lipschitz continuous by (5.1.1). That is we can evaluate the above terms summing the square root of the mean squared error studied before. Indeed (7.2.2) is bounded with the above Lemma 5.2.2. Lemma 5.2.5 is used for the bound of (7.2.3). The bound for (7.2.4) is studied in Lemma 7.2.3. Finally, (7.2.5) is straightforwardly bounded by $C_{\kappa,p}(T/\sqrt{n})^{(p-1)/p}$.

The proof of Theorem 7.1.1 is achieved.

Chapter 8

Integrability of S and various lemmata

8.1 Integrability of S

We shall show that under Assumption 5.1.2, the moments of S and S^κ exist. We recall that we write

$$dY_t = \sigma(t, S_t)dW_t + r_t dt - \frac{1}{2}\sigma^2(t, S_t)dt.$$

First note that if the function r is bounded and $\sigma^2(t,x) \leq L(1+\ln(\ln(x))1_{x>1})$, we deduce that for all $t \in [0,T]$

$$E \sup_{u \leq t} Y_u^2 \leq C + C \int_0^t E \sup_{u \leq r} Y_u^2 dr < \infty.$$

By Gronwall's Lemma, we deduce that $E \sup_{u \leq T} Y_u^2 < \infty$. Hence the process $\int_0^{\cdot} \sigma(t, S_t)dW_t$ is a true martingale.

Lemma 8.1.1 *Assume that Assumption 5.1.2 holds, then there exists a constant C independent of κ such that* $\sup_{u \leq T} ES_u^\kappa \leq C$.

Proof. In view of the definition of S^κ in Paragraph 5.2.1, we have $S_u^\kappa / F(u) \leq M_u$ where $M_0 = 1$ and M is the local martingale solution to the s.d.e.

$$dM_u = M_u \sigma(u, S_u^\kappa \wedge \kappa) dW_u.$$

As r is bounded, we get $S_u^\kappa \leq C_x M_u$. Using a sequence of stopping times (τ^n) such that M^{τ^n} is a true martingale, we deduce that $ES_{u\wedge\tau^n}^\kappa \leq Cx$ and by Fatou's lemma we get that $ES_u^\kappa \leq Cx$. □

Lemma 8.1.2 *Assume that Assumption 5.1.2 holds, then for $p > 1$ there exists a constant C_p independent of κ such that* $\sup_{u\leq T} E(S_u^\kappa)^p \leq C_p$.

Proof. We have, with q to be defined later,

$$(S_u^\kappa)^p \leq CxN_u e^{\frac{1}{2}(q-p)\int_0^u \sigma^2(u, S_u^\kappa \wedge\kappa)du},$$

where

$$N_u := \exp\left(p\int_0^u \sigma(r, S_r^\kappa \wedge \kappa)dW_r - \frac{1}{2}q\int_0^u \sigma^2(r, S_r^\kappa \wedge \kappa)dr\right).$$

Using the inequality $0 \leq ab \leq (a^2 + b^2)$, we get that

$$(S_u^\kappa)^p \leq Cx\tilde{N}_u + Cxe^{(q-p)\int_0^u \sigma^2(r, S_r^\kappa\wedge\kappa)dr},$$

where $\tilde{N} = N^2$ is a local martingale when choosing $q = 2p^2$. Moreover as the function $x \mapsto e^{(q-p)ux}$ is convex, the Jensen inequality and the hypothesis yields

$$\begin{aligned} e^{(q-p)\int_0^u \sigma^2(u, S_u^\kappa\wedge\kappa)du} &\leq \frac{1}{u}\int_0^u e^{(q-p)s\sigma^2(s, S_s^\kappa\wedge\kappa)}ds \\ &\leq C_p + \frac{1}{u}\int_0^u \left(\log(S_s^\kappa \vee 1)\right)^{k(q-p)T} ds, \end{aligned}$$

where k is a constant. Using the property

$$\left(\log(x \vee 1)\right)^{k(q-p)T} \leq C_p x, \quad \forall x \geq 0,$$

and Lemma 8.1.1, we deduce that $E(S_u^\kappa)^p \leq C_p$. □

Corollary 8.1.3 *Assume that Assumption 5.1.2 holds, then there exists a constant C_p independent of κ such that* $E\sup_{u\leq T}(S_u^\kappa)^p \leq C_p$.

Proof. We may assume without loss of generality that $p \in 2\mathbb{N}$. We recall that

$$S_t^\kappa = x + \int_0^t \sigma(u, S_u^\kappa \wedge \kappa) S_u^\kappa dW_u + \int_0^t r_u S_u^\kappa du.$$

Since $0 \leq \sigma^2(t,x) \leq C(1+x)$ we deduce easily that $E \sup_{u \leq T} (S_u^\kappa)^p \leq C_p$ by using the Burkholder–Davis–Gundy inequalities and Lemma 8.1.2. □

Since S^κ converges pointwise on $[0,T]$ to S, we deduce the following with Fatou's Lemma.

Lemma 8.1.4 *Assume that Assumption 5.1.2 holds, then there exist constants C_p such that $E \sup_{u \leq T} (S_u)^p \leq C_p$.*

8.2 Various Lemmata

This section gathers a few technical lemmata.

Lemma 8.2.1 *Let $(g^m)_m$ be a sequence of positive functions defined on an interval $[0,T]$, $T > 0$ such that for some $C > 0$, we have:*

$$g^{m+1}(t) \leq C \int_0^t g^m(u) du, \quad 0 \leq g^0 \leq C.$$

Then

$$\sup_{t \in [0,T]} g^m(t) \leq C^m \frac{t^m}{m!} \sup_{t \in [0,T]} g^0(t).$$

Proof. The proof stands on the following induction. Set

$$C_g := \sup_{t \in [0,T]} g^0(t) \leq C.$$

Suppose that

$$\sup_{t \in [0,T]} g^{n-1}(t) \leq C^{n-1} \frac{t^{n-1}}{(n-1)!} C_g,$$

we have

$$g^n(t) \leq C \int_0^t C^{n-1} \frac{s^{n-1}}{(n-1)!} C_g ds.$$

And the result is stated. □

Lemma 8.2.2 *Let $(g^m)_m$ be a sequence of positive functions defined on an interval $[0,T]$, $T>0$ such that g_1 is bounded and, for some C, $\overline{C}>0$, we have:*

$$g^{m+1}(t) \leq C\int_0^t g^m(u)du + \overline{C}.$$

Then

$$\sup_m \sup_{t\in[0,T]} g^m(t) \leq \max(\sup_{t\in[0,T]} g^1(t); \overline{C})e^{C(1+T)}.$$

Proof. Set $K = \max(\sup_{t\in[0,T]} g^1(t); \overline{C})$. The result is proved by induction. That is, $g^1(t) \leq K\exp(Ct)$. Suppose that $g^n(t) \leq K\exp(Ct)$, we have

$$g^{m+1}(t) \leq C\int_0^t K\exp(Cu)du + K = K\exp(Ct).$$

Which yields the result.□

Lemma 8.2.3 *Let $a_1, \cdots a_k$ be real numbers. We have the following inequalities*

$$\left(\sum_{l=1}^k a_l\right)^2 \leq \sum_{l=1}^k 2^l a_l^2, \qquad \left(\sum_{l=1}^k a_l\right)^4 \leq \sum_{l=1}^k 8^l a_l^4.$$

Proof. For any real numbers a, b, the inequalities

$$2ab \leq a^2 + b^2,$$

leads to

$$(a+b)^2 \leq 2a^2 + 2b^2,$$

which leads to

$$(a+b)^4 \leq (2a^2+2b^2)^2 \leq 8a^4 + 8b^4.$$

We show the results by evident inductions. □

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Les aspects mathématiques des modèles de marchés financiers avec coûts de transaction

Les marchés financiers occupent une place prépondérante dans l'économie. La future évolution des législations dans le domaine de la finance mondiale va rendre inévitable l'introduction de frictions pour éviter les mouvements spéculatifs des capitaux, toujours menaçants d'une crise. C'est pourquoi nous nous intéressons principalement, ici, aux modèles de marchés financiers avec coûts de transaction.

Cette thèse se compose de trois chapitres. Le premier établit un critère d'absence d'opportunité d'arbitrage donnant l'existence de systèmes de prix consistants, i.e. martingales évoluant dans le cône dual positif exprimé en unités physiques, pour une famille de modèles de marchés financiers en temps continu avec petits coûts de transaction.

Dans le deuxième chapitre, nous montrons la convergence des ensembles de sur-réplication d'une option européenne dans le cadre de la convergence topologique des ensembles. Dans des modèles multidimensionnels avec coûts de transaction décroissants a l'ordre $n^{-1/2}$, nous donnons une description de l'ensemble limite pour des modèles particuliers et en déduisons des inclusions pour les modèles généraux (modèles de KABANOV).

Le troisième chapitre est dédié a l'approximation du prix d'options européennes pour des modèles avec diffusion très générale (sans coûts de transaction). Nous étudions les propriétés des pay-offs pour pouvoir utiliser au mieux l'approximation du processus de prix du sous-jacent par un processus intuitif défini par récurrence grâce aux itérations de PICARD.

Mathematical Aspects of Financial Market Models with Transaction Costs

Financial markets play a prevailing role in the economy. The future legislation development in the field of global finance will unavoidably lead to friction to prevent speculative capital movements, always threatening with crisis. That is why we are interested in the financial market models with transaction costs.

This thesis consists of three chapters. The first one establishes a criterion of absence of arbitrage opportunities giving the existence of consistent price systems, i.e. martingale evolving in the dual cone expressed in physical units. The criterion holds for a family of financial market models in continuous time with small transaction costs.

In the second chapter, we show the convergence of super-replication sets for a European option in the context of the topological convergence of sets. In multivariate models with transaction costs decreasing at rate $n^{-1/2}$, we give a description of the limit set for specific models. We deduce inclusions for general models (KABANOV's models).

The third chapter is dedicated to the approximation of the European option price for models with very general diffusion (without transaction costs). We study properties of the pay-off to make best use of the approximation of the underlying asset price, based on PICARD iterations.

Keywords : Transaction Costs, Multidimensional Models, European Option, Arbitrage Theory, Super-Replication, Topological Convergence, Diffusion Process.

Mathematics Subject Classification (2000) 60G44.